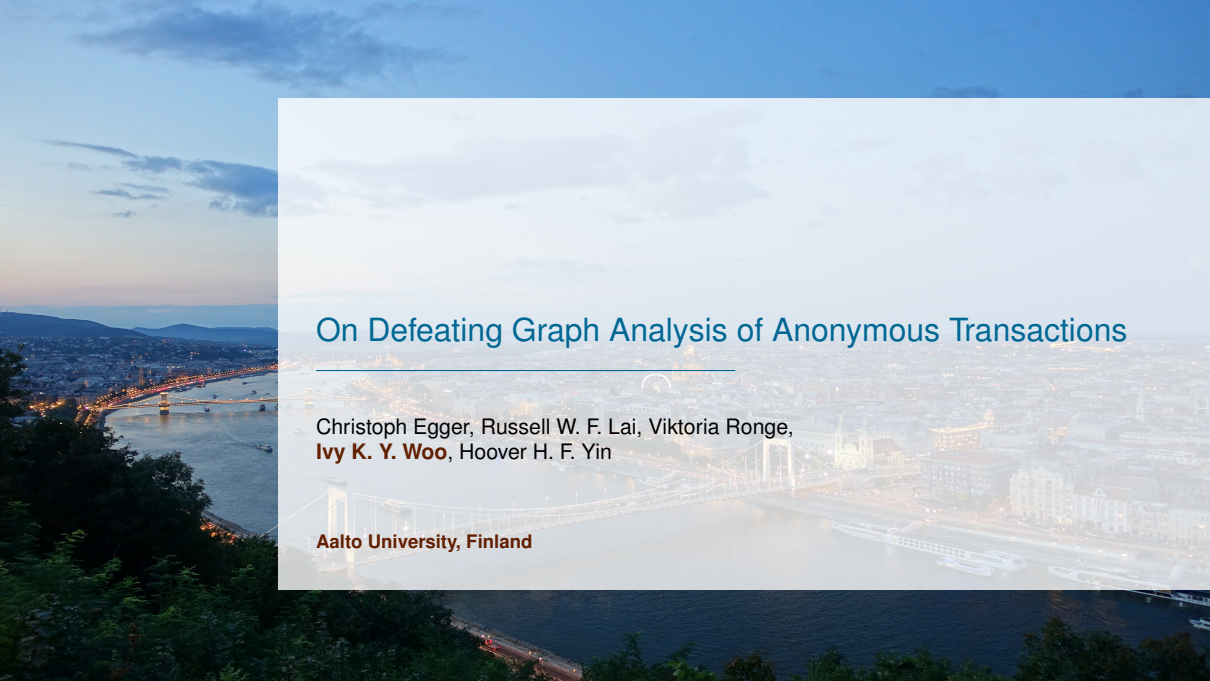

On Defeating Graph Analysis of Anonymous Transactions

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Ivy K. Y. Woo, Hoover H. F. Yin

Aalto University, Finland

Content

Content

The Example of Anonymous Cryptocurrencies

Alice

Blockchain/Everyone

Choose decoys .

"I own one among (ring)

tx := which contains 1 coin, and

I'm transferring 1 coin to Bob's ."

$\pi \leftarrow$ Prove(tx)

$\text{tx}, \pi \longrightarrow$

Ring Samplers

- $U :=$ set of all users
- $\mathcal{R} \leftarrow \text{Samp}(i)$:
  PPT algorithm, inputs signer $i \in U$, outputs a ring $\mathcal{R}$ such that $i \in \mathcal{R} \subseteq U$
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- But wait, how to quantify "anonymity" in the first place?
  (Not simply $1/|\mathcal{R}|$, since users could have different probability to be the real signer.)

Entropy-based Anonymity [Ronge et al. (2021)]

- Let $\mathcal{S}$ be signer distribution, $\mathcal{R} := \text{Samp}(\mathcal{S})$ be ring sampled.
  Anonymity of a ring sampler measured by conditional min-entropy:

$$H_\infty(\mathcal{S}|\mathcal{R}) = -\lg(\text{Guess}(\mathcal{S}|\mathcal{R})),$$

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- Higher $H_\infty(\mathcal{S}|\mathcal{R}) \Leftrightarrow$ More difficult to guess the real signer given the sampled ring
- Can be viewed as “local” anonymity of ring samplers
  - Does not take into account how other users sample their rings

Some Classes of Ring Samplers

1. Uniform Samplers
   - Sample k decoys uniformly randomly from set of all users
   - Do not take signer distribution $\mathcal{S}$ into account
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2. Mimicking Samplers
   - Sample k decoys according to signer distribution $\mathcal{S}$
   - Near-optimal anonymity
   - Problem: requires to know signer distribution, which is hard in reality

Some Classes of Ring Samplers

3. Partitioning Samplers
   - Partition the set of all users into chunks, sample k decoys uniformly randomly from the chunk that the signer belongs to
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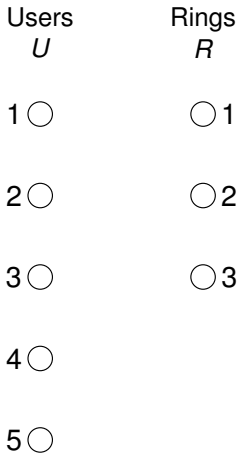
Question

How about “global” anonymity?
How well can partitioning samplers resist “global” attacks (for certain ring size), e.g. attacks based on graph-analysing ring memberships of all transactions?

Content

Transaction Graph

- Bipartite graph $G = (U, R, E)$
- Nodes U: set of users
- Nodes $R \subseteq U$: set of rings



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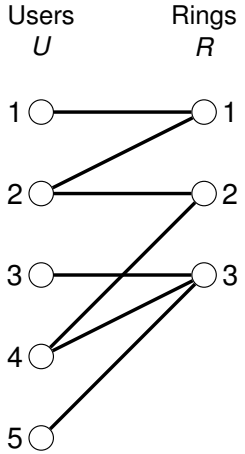

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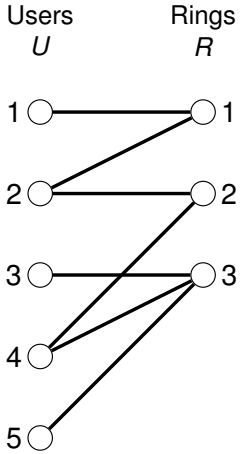

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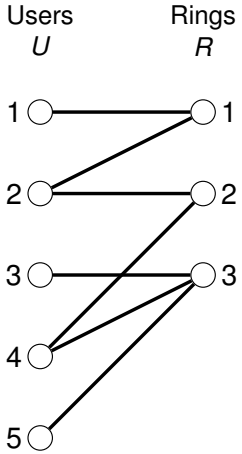

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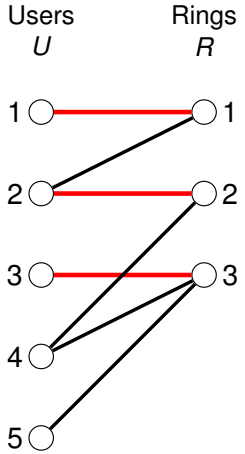

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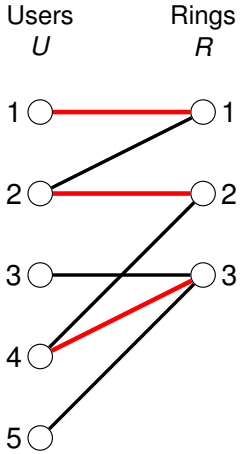

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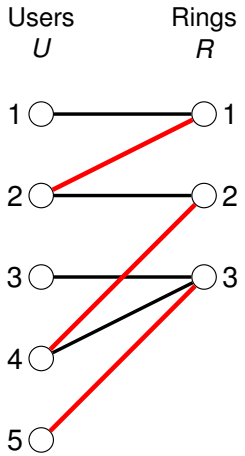

Modelling Attack by Security Game

$$\boxed{\begin{array}{l}\mathsf{Exp}_{\mathcal{A},\mathsf{Samp}}(U,|R|)\\ \hline (G=(U,R,E),M)\leftarrow \mathcal{G}^{\mathsf{Samp}}(U,|R|)\\ (u^*,r^*)\leftarrow \mathcal{A}(G)\\ \textbf{return } ((u^*,r^*)\in M)\end{array}}$$

- $U =$ set of users, $R =$ set of rings, $G =$ transaction graph,
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- $\mathcal{G}^{\mathsf{Samp}}$: takes ring sampler Samp as oracle and samples (G, M)
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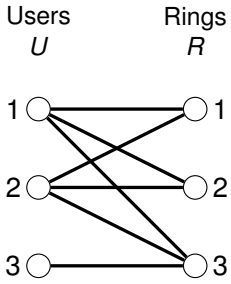
- Given G, adversary $\mathcal{A}$ wins if it outputs an edge in M
- Lower $\Pr\left[\mathsf{Exp}_{\mathcal{A},\mathsf{Samp}}(U,|R|)\right] \Leftrightarrow$ Higher anonymity under graph-based attacks

What can Adversary do with G?

Given transaction graph $G = (U, R, E)$,

- Dulmage-Mandelsohn(DM) decomposition
  - Core $(G) = (U, R, E')$, where $E' \subseteq E$ is union of all maximum matchings



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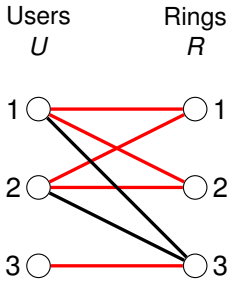

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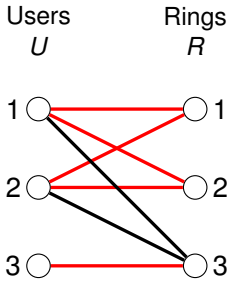

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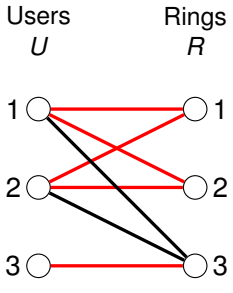

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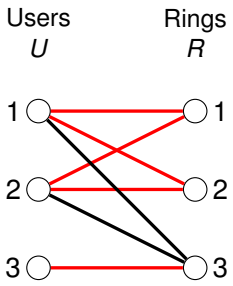
  - $G \neq$ Core $(G) \Rightarrow$ Decreased anonymity



Upper-bounding $\Pr\left[\mathrm{Exp}_{\mathcal{A},\mathrm{Samp}}(U,|R|)\right]$

- A graph-analysing adversary can exclude edges in G not in $\mathrm{Core}(G)$

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Relation between $\Pr_{G\leftarrow\mathcal{G}^{\mathrm{Samp}}}[G \neq \mathrm{Core}(G)]$ and $\Pr\left[\mathrm{Exp}_{\mathcal{A},\mathrm{Samp}}(U,|R|)\right]$?

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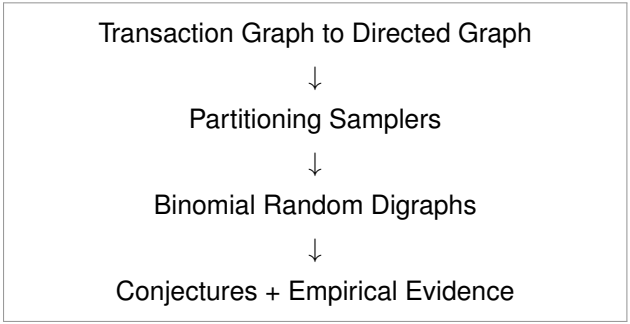
Question 2

How to upper-bound $\Pr_{G\leftarrow\mathcal{G}^{\mathrm{Samp}}}\left[G\neq\mathrm{Core}\,(G)\right]$ for $\mathrm{Samp}=$ partitioning sampler?

Roadmap


Transaction Graph to Directed Graph
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Binomial Random Digraphs
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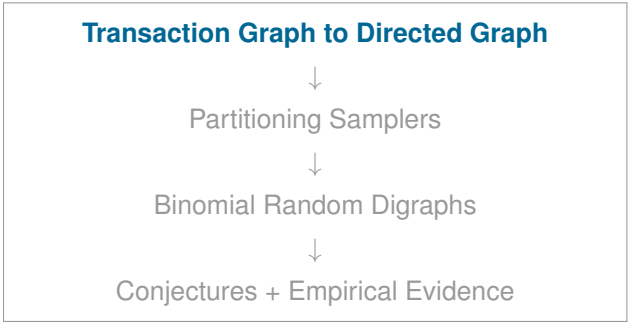
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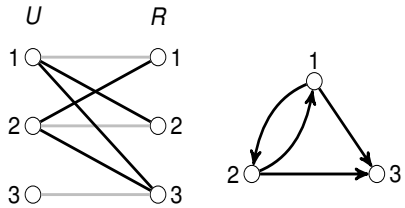
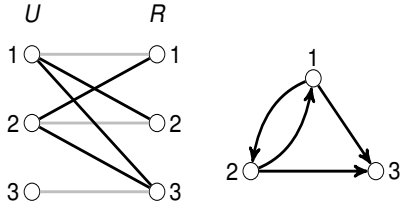
Induced Directed Graph



- Transaction graph $G = (U, R, E)$
- Induced digraph of G:
  $\mathrm{id}(G) = (U, F)$ where $F = \{(i, j) : (i, j) \in E, i \neq j\}$
- Biadjacency matrix of $G \approx$ Adjacency matrix of $\mathrm{id}(G)$

Finding Core (G)

Case 1: $|U| = |R|$ (balanced)



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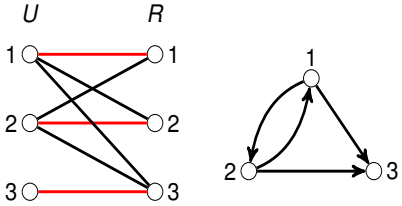

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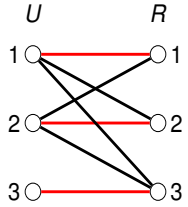

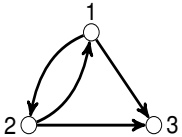

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$$(i, j) \text{ is an edge of a strongly connected component in id } (G)$$
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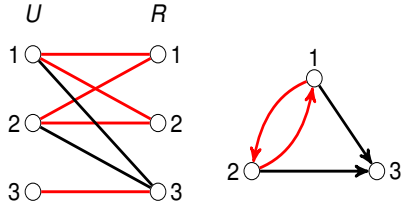

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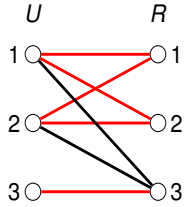

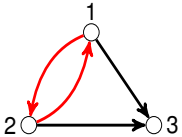

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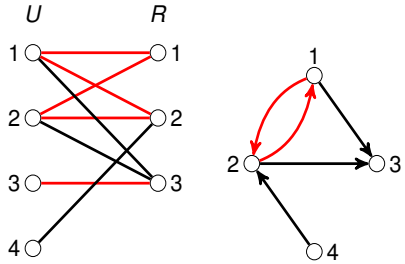
$\Rightarrow$ If id (G) is strongly connected (S.C.), then $G = \text{Core } (G)$

$\Rightarrow \Pr_{G \leftarrow \mathcal{G}} [G \neq \text{Core } (G)] \leq \Pr_{G \leftarrow \mathcal{G}} [\text{id } (G) \text{ not S.C.}]$

Finding Core (G)

Case 2: $|U| > |R|$ (imbalanced)



Finding Core (G)

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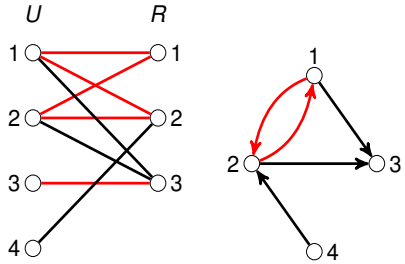
- Lower nodes = $\{i : i \in U, i > |R|\}$



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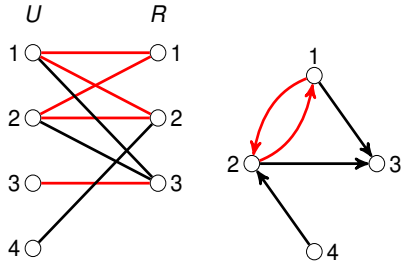

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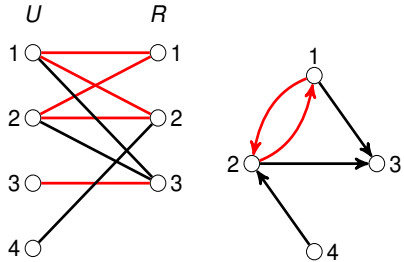

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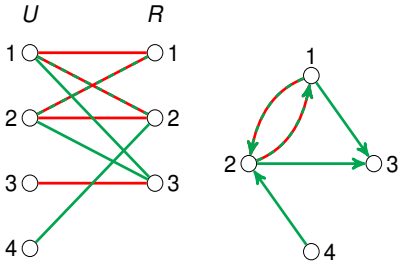

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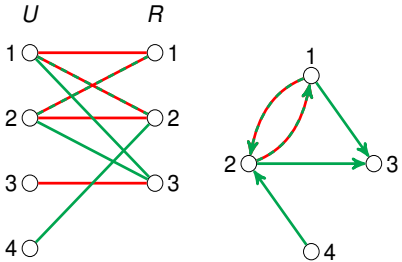

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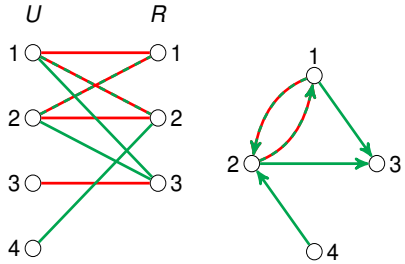
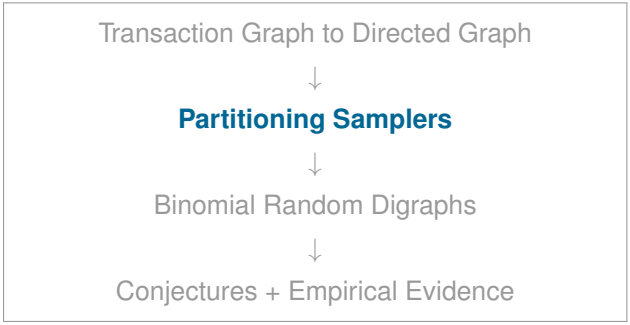


- Intuitively, the existence of lower nodes can significantly increases $\Pr[G = \text{Core}(G)]$
- We proved that $\Pr[G \neq \text{Core}(G)]$ is greatest when $|R| = |U|$ (for fixed $|U|$)

Roadmap


Transaction Graph to Directed Graph
Partitioning Samplers
Binomial Random Digraphs
Conjectures + Empirical Evidence

Induced Digraphs of Partitioning Samplers

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  $\rightarrow$ The Scottish Book (problem 38): Open problem for over 40 years...?

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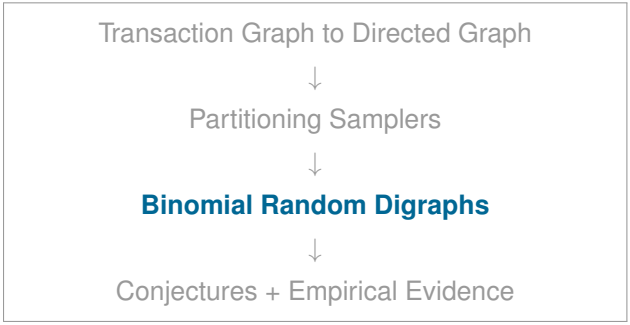
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  Graham and Pike (2008):

  If $p = p(n) = \frac{\ln n + c}{n}$ for some constant $c \in \mathbb{R}$,

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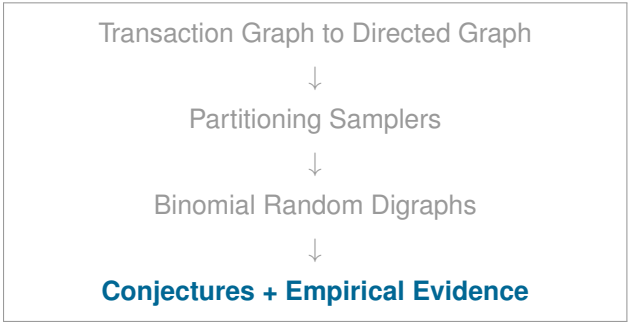
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$\mathcal{G}^{\mathtt{reg}}$: Fix in-degree $= k$ for each node

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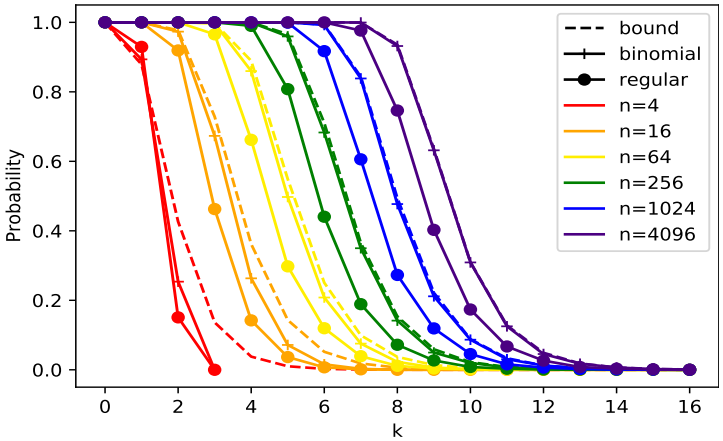

Figure: Pr [G not S.C.] against k under various n.

Chaining Things Together

Consider transaction graph G of a chunk. Let n be chunk size, k be number of decoys.

$$
\begin{aligned}
& \Pr_{G \leftarrow \mathcal{G}^{\mathtt{Samp}}} [G \neq \mathrm{Core}(G)] \quad \text{for any } |R| \leq n \\
\leq & \Pr_{G \leftarrow \mathcal{G}^{\mathtt{Samp}}} [G \neq \mathrm{Core}(G)] \quad \text{for } |R| = n \\
\leq & \Pr_{G \leftarrow \mathcal{G}^{\mathtt{Samp}}} [\mathrm{id}(G) \text{ not S.C.}] \\
= & \Pr_{G \leftarrow \mathcal{G}^{\mathtt{reg}}} [G \text{ not S.C.}] \\
\leq & \Pr_{G \leftarrow \mathcal{G}^{\mathtt{bin}}} [G \text{ not S.C.}], \; p = \tfrac{k}{n-1} \qquad \text{(Conj.1)} \\
\leq & 1 - e^{-2e^{\ln n - \frac{k}{n-1} n}} \qquad \text{(Conj.2)}
\end{aligned}
$$

$\Pr[G \neq \mathrm{Core}(G)]$ for the set of all users: apply union bound.

Implication

- Recall:

$$\Pr\left[\mathrm{Exp}_{\mathcal{A},\mathrm{Samp}}(U,|R|)\right] \leq \Pr_{G\leftarrow\mathcal{G}^{\mathrm{Samp}}}\left[G \neq \mathrm{Core}\left(G\right)\right] + \frac{1}{k+1}$$

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- If Conjectures 1 and 2 hold, it suffices to set
$$k \geq \ln(2|U|) + \sqrt{2\ln(2|U|)},$$
i.e. set k to be logarithmic in number of users to resist graph-based attacks.

Summary

- Background of ring-signature-based anonymous systems and ring samplers
- Modelling anonymity of ring samplers by transaction graphs and security games
- Conjectures about strong connectivity of random directed graphs
- Provably secure ring size for partitioning samplers to resist graph-based deanonymisation attacks
- E-print: `ia.cr/2022/132`

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Thank You!

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