Modern tontine with bequest: Innovation in pooled annuity products

Thomas Bernhardt and Catherine Donnelly (2019).
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1 Introduction

Due to increasing human life expectancy and tightening regulatory requirements, insurance companies are facing increasing pressure from risk capital requirements for providing retirement products such as annuities, and retirees in turn have to bear an increasing risk capital cost embedded in the insurance premium. In recent years, in attempt to alleviate the problem, various product structures aiming at pooling mortality risk of retirees have been investigated in the literature, one of which is tontine (see e.g. Milevsky and Salisbury (2015, 2016) and Chen et al. (2019)).

In this paper the authors introduce a new retirement product design, namely by incorporating a bequest account to the tontine structure that is known in literature. The wealth in the bequest account is to be left to the retiree's family after the retiree's death. This contrasts with the traditional tontine design that, a participant of tontine must give up all of her wealth at death. Under the CRRA utility maximization framework, the authors have solved the optimal strategies for (1) investment in the financial market which is assumed to consist of one risk-free asset and one risky asset, (2) investment in the tontine account and (3) consumption for a participant of the proposed product.

In this seminar, we are going to look at the simplified version, in which there is only one risk-free asset in the financial market, and hence investment return from the financial market is always the constant risk-free rate.

2 Product Mechanism

There exist two accounts in the product structure:

- a **tontine account**, the account value of which
  - attracts mortality credits, and
  - is to be shared among other surviving participants in case of death;
- a **bequest account**, the account value of which
  - does not attract mortality credits, and
  - is to be left to the family of the retiree in case of death.

In addition, both accounts earn investment returns at the same rate, and consumption is to be withdrawn from both accounts at the same rate.

A special feature of the proposed design is called **re-balancing**. This refers to the mechanism that, after mortality credits are attracted by the tontine account, they are to be shared among the two accounts, such that the ratio of the two account values always remains constant. In the following we denote by the constant $\alpha \in [0, 1]$ the portion of wealth in the tontine account, and $1 - \alpha$ is the portion of wealth in the bequest account.

Model Set-up

We consider an infinite pool of homogeneous participants for the tontine with bequest product. It is assumed that the mortality rate of participants is deterministic. We set time 0 as the retirement age 65. The mortality rate of participants follows the Gompetz-Makeham law, which has the form $\lambda(t) = A + B \cdot C^{65+t}$ where A, B and C are some constants.

Denote by $X(t)$ the total account value at time t. The dynamics of $X(t)$ is

$$\mathrm{d}X(t) = (r + \alpha\lambda(t) - c(t))X(t)\mathrm{d}t, \quad X(0) > 0 \text{ constant}, \tag{1}$$

where $r > 0$ is the risk-free return rate and $c(t)$ is the consumption rate at time t.

3 Utility Maximization

We consider the power utility

$$U(x) = \frac{x^\gamma}{\gamma}, \quad \gamma \in (-\infty, 1)\backslash\{0\},\ x \geq 0,$$

where $1 - \gamma$ is the relative risk aversion level.

Our aim is to maximize the expected lifetime utility of an individual participating in the tontine with bequest, which is

$$\mathbb{E}\left[\int_0^\tau e^{-\rho s}\frac{(c(s)X(s))^\gamma}{\gamma}\mathrm{d}s + be^{-\rho\tau}\frac{((1-\alpha)X(\tau))^\gamma}{\gamma}\right] =: \mathbb{E}\left[\mathcal{U}\right], \tag{2}$$

where τ denotes the random remaining lifetime of the concerned individual, ρ is the constant time preference rate on consumption and b is constant representing the individual's strength of bequest motive.

3.1 Variable consumption rate $c(t)$

Suppose that $\alpha \in [0, 1]$ is already fixed by the retiree. We want to look for the optimal consumption rate $c^\star(t)$ which maximizes (2), subject to (1).

Here $c^\star(t)$ is found by solving the corresponding Hamilton-Jacobi-Bellman (HJB) equation (for reference on the technique see e.g. Björk (2009)). The solution is given as

$$c^\star(t) = \big(\gamma h(t)\big)^{\frac{1}{\gamma-1}},$$

where $h(t)$ is the solution to

$$0 = \partial_t h(t) + (1-\gamma)\gamma^{\frac{1}{\gamma-1}}h(t)^{\frac{\gamma}{\gamma-1}} + h(t)\psi(t) + \varphi(t)$$

with

$$\psi(t) := \gamma\big(r + \alpha\lambda(t)\big) - \lambda(t) - \rho, \quad \varphi(t) := \frac{b}{\gamma}\lambda(t)(1-\alpha)^\gamma,$$

and the terminal condition for $h(t)$ being $\lim_{t\to\infty} h(t) = b\frac{(1-\alpha)^\gamma}{\gamma}$.

3.2 Constant consumption rate c

Now suppose consumption rate is constant, so $c(t) = c$. We want to look for the optimal strategies $\alpha^\star$ and $c^\star$ which together maximize (2), subject to (1), $\alpha \in [0, 1]$ and $c \geq 0$.

With some algebra, the expected lifetime utility (2) can be written as

$$\mathbb{E}\left[\mathcal{U}\right] = \begin{cases} \frac{X(0)^\gamma}{\gamma}\left(b\frac{(1-\alpha)^\gamma}{1-\gamma\alpha}M_{A_\gamma}(-k) + c^\gamma\frac{1-M_{A_\gamma}(-k)}{k}\right), & \text{if } k \neq 0 \\ \frac{X(0)^\gamma}{\gamma}\left(b\frac{(1-\alpha)^\gamma}{1-\gamma\alpha} + c^\gamma\mathbb{E}\left[A_\gamma\right]\right), & \text{if } k = 0 \end{cases} \tag{3}$$

where A_γ is defined as a random variable with tail distribution $\mathbb{P}(A_\gamma > t) = \mathbb{P}(\tau > t)^{1-\gamma\alpha}$, $k := \gamma(c - r) + \rho$ and $M_{A_\gamma}(-k) = \mathbb{E}\left[e^{-kA_\gamma}\right]$.

As there is no analytical solution for $\alpha^\star$ and $c^\star$ available, they are solved numerically by evaluating (3).

Discussion

For the numerical illustration, I follow the paper to set mortality rate as $\lambda(t) = 2.2 \cdot 10^{-4} + 2.7 \cdot 10^{-6} \cdot 1.124^{65+t}$, and I set $r = 5\%$ and $\rho = 3\%$. The following observations for $\alpha^\star$ and $c^\star$ are obtained:

- $\alpha^\star$ and $c^\star$ are always higher for those with smaller bequest motive.
- $c^\star$ is increasing with the relative risk aversion level. It reaches a stable rate of around 8.5% for all values of b's when relative risk aversion level $1 - \gamma > 2$.
- Comparing with the case when there is no tontine available (i.e. $\alpha = 0$), the optimal consumption rate with tontine available is higher in most cases.
- $\alpha^\star$ is increasing with the relative risk aversion level, except when the relative risk aversion level is very small ($1 - \gamma < 0.5$), where $\alpha^\star$ exhibits an opposite trend.

  This opposite trend is explained by the very large bequest account value obtained at old age if retiree survives until then. This explanation is verified twofold, by (1) removing utility from old age and (2) projecting the bequest account value until old age.

4 Including risky investment

In the paper, the authors consider a financial market which consists of one risk-free asset earning risk-free rate r and one risky asset earning a mean return rate μ with volatility σ. A retiree invests $\omega(t)$ in the risky asset and $1 - \omega(t)$ in the risk-free asset at time t. The authors have solved for the optimal investment strategy $\omega^\star(t)$, in addition to $\alpha^\star$ and $c^\star(t)$ (both of the cases where $c(t)$ is variable with t and $c(t)$ is constant).

It is found that as long as $c(t)$ is deterministic, $\omega^\star(t)$ is always a constant independent of α and $c(t)$. For $\alpha^\star$ and $c^\star(t)$, similar observations have been obtained as in the risk-free cases discussed above.

The results when considering also the risky asset are not to be discussed in this seminar, but a summary of the results from the paper is given here as reference for those who are interested.

4.1 Variable consumption rate $c(t)$

Suppose $\alpha \in [0, 1]$ is already fixed by the retiree. We want to look for the optimal strategies $\omega^\star(t)$ and $c^\star(t)$ which together maximize (2), subject to (1).

The optimal strategies $\omega^\star(t)$ and $c^\star(t)$ are found by solving the corresponding HJB equation. The solutions are

$$\omega^\star(t) = \omega^\star = \frac{1}{1-\gamma} \frac{\mu - r}{\sigma^2}$$

and

$$c^\star(t) = \big(\gamma h(t)\big)^{\frac{1}{\gamma-1}},$$

where $h(t)$ is the solution to

$$0 = \partial_t h(t) + (1-\gamma)\gamma^{\frac{1}{\gamma-1}} h(t)^{\frac{\gamma}{\gamma-1}} + h(t)\psi(t) + \varphi(t)$$

with

$$\psi(t) := \gamma\big(r + \alpha\lambda(t)\big) + \frac{1}{2}\frac{\gamma}{\gamma-1}\left(\frac{\mu - r}{\sigma}\right)^2 - \lambda(t) - \rho, \quad \varphi(t) := \frac{b}{\gamma}\lambda(t)(1-\alpha)^\gamma.$$

Note that the only difference for the solution of $c^\star(t)$ here comparing to the risk-free case lies in the function $\psi(t)$.

In order to find $c^\star(t)$ from the formulas above, a transversality condition for $h(t)$ has to be determined. This has not been discussed in details in the paper and no numerical result has been run, although it is believed that the same terminal condition as in the risk-free case could be used here.

4.2 Constant consumption rate c

Since it has been proved in the paper that $\omega^\star(t)$ is independent of α and c, we can solve for $\alpha^\star$ and $c^\star$ as follows.
Step 1: Solve for $\omega^\star(t)$, which is found to be a constant

$$\omega^\star(t) = \omega^\star = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}. \tag{4}$$

Step 2: Substitute (4) into (2) to obtain a new formula for the expected lifetime utility, which after some algebra can be written as

$$\mathbb{E}\left[\mathcal{U}\right] = \begin{cases} \frac{X(0)^\gamma}{\gamma}\left(b\frac{(1-\alpha)^\gamma}{1-\gamma\alpha}M_{A_\gamma}(-k) + c^\gamma \frac{1-M_{A_\gamma}(-k)}{k}\right), & \text{if } k \neq 0 \\ \frac{X(0)^\gamma}{\gamma}\left(b\frac{(1-\alpha)^\gamma}{1-\gamma\alpha} + c^\gamma \mathbb{E}\left[A_\gamma\right]\right), & \text{if } k = 0 \end{cases} \tag{5}$$

where A_γ is defined as a random variable with tail distribution $\mathbb{P}(A_\gamma > t) = \mathbb{P}(\tau > t)^{1-\gamma\alpha}$, $k := \frac{1}{2}\frac{\gamma}{\gamma-1}\left(\frac{\mu-r}{\sigma}\right)^2 + \gamma(c-r) + \rho$ and $M_{A_\gamma}(-k) = \mathbb{E}\left[e^{-kA_\gamma}\right]$.
Note that the only difference here comparing to the risk-free case lies in the definition of k.
Step 3: Solve for $\alpha^\star$ and $c^\star$ numerically by evaluating (5), subject to $\alpha \in [0,1]$ and $c \geq 0$.

References

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