Seminar - Special Aspects of Insurance Economics

Modern tontine with bequest: Innovation in pooled annuity products

T. Bernhardt and C. Donnelly

Ivy Woo | December 13, 2019 | Institute of Insurance Science

Content

Content

What is Tontine?

- Participants contribute money to the pool at inception
- Total benefit to the pool at each payment time is **predetermined**, but shared only among **surviving** members
- Traditional Tontine - example:
  260 participants, each contributes €400, interest 25%

  $\rightarrow$Interest $= 260 \times 400 \times 25\% =$ €26000 split among surviving members
  - all survive: each receives €100
  - 250 survive: each receives €104
    In other words:
    €100 interest + €4 payout from **mortality credits**

What is Tontine?

- Participants contribute money to the pool at inception
- Total benefit to the pool at each payment time is **predetermined**, but shared only among **surviving** members
- Traditional Tontine - example:
  260 participants, each contributes €400, interest 25%
  $\rightarrow$Interest $= 260 \times 400 \times 25\% =$ €26000 split among surviving members
  - all survive: each receives €100
  - 250 survive: each receives €104

  In other words:
  €100 interest + €4 payout from **mortality credits**

What is Tontine?

- Participants contribute money to the pool at inception
- Total benefit to the pool at each payment time is **predetermined**, but shared only among **surviving** members
- Traditional Tontine - example:
  260 participants, each contributes €400, interest 25%
  $\rightarrow$Interest $= 260 \times 400 \times 25\% =$ €26000 split among surviving members
  - all survive: each receives €100
  - 250 survive: each receives €104
    In other words:
    €100 interest + €4 payout from **mortality credits**

Why Tontine?

- Mortality risk: uncertainty on the death time of an individual
- Tontine **pools mortality risk** of a group of people
- Insurance company is exposed to less mortality risk, as the participants share part of the risk among themselves
  - Insurance company: less risk capital required
  - Policyholders: bear additional mortality risk, in exchange for less risk capital charges included in the gross premium of product

Questions

- Other payment structures?
- Optimal payments under utility maximization framework?

⇒ Some other papers:

- Milevsky, M. A., & Salisbury, T. S. (2015). Optimal retirement income tontines. Insurance: Mathematics and economics, 64, 91-105.
- Milevsky, M. A., & Salisbury, T. S. (2016). Equitable retirement income tontines: Mixing cohorts without discriminating. ASTIN Bulletin, 46(3), 571-604.

Content

How does it work?

- Tontine account:
  - attracts mortality credits
  - shared among other surviving participants in case of death
- Bequest account:
  - not attract mortality credits
  - left to family in case of death
- Both accounts earn investment return at same rate
- Consumption is withdrawn from both accounts at same rate

Re-balancing

- Suppose the initial ratio of the two account values is:

  α in Tontine account, $1 - \alpha$ in Bequest account, $\alpha \in [0, 1]$.

- Mortality credits, after received, are shared among the two accounts so that the ratio of the two account values is kept constant.

  $\Rightarrow \alpha$ into Tontine account, $1 - \alpha$ into Bequest account.

Assumptions

- Mortality rate is deterministic, same for all participants and follows the Gompertz-Makeham law:

$$\lambda(t) = A + B \cdot C^{65+t}$$

- Infinite pool size
  $\Rightarrow$Participants perishing at rate $\lambda(t)$

Wealth dynamics

$Y(t) :=$ value of tontine account at time t

$Z(t) :=$ value of bequest account at time t

$X(t) := Y(t) + Z(t) =$ total account value at time t

Suppose there is investment in a risk-free asset.
Account value dynamics:

- $\mathrm{d}X(t) = (r + \alpha\lambda(t) - c(t))X(t)\mathrm{d}t,\ X(0) > 0$ constant
- $\mathrm{d}Y(t) = \alpha\mathrm{d}X(t) = (r + \alpha\lambda(t) - c(t))Y(t)\mathrm{d}t,\ Y(0) = \alpha X(0)$
- $\mathrm{d}Z(t) = (1-\alpha)\mathrm{d}X(t) = (r+\alpha\lambda(t)-c(t))Z(t)\mathrm{d}t,\ Z(0) = (1-\alpha)X(0)$

r : risk-free rate, $c(t)$: consumption rate

Wealth dynamics

$$Y(t) := \text{value of tontine account at time } t$$

$$Z(t) := \text{value of bequest account at time } t$$

$$X(t) := Y(t) + Z(t) = \text{total account value at time } t$$

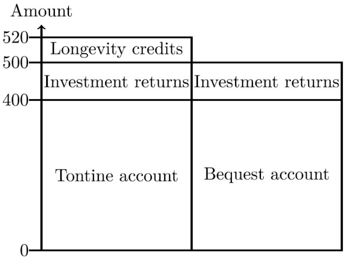
Suppose there is investment in a risk-free asset.
Account value dynamics:

- $\mathrm{d}X(t) = (r + \alpha\lambda(t) - c(t))X(t)\mathrm{d}t, \ X(0) > 0$ constant
- $\mathrm{d}Y(t) = \alpha\mathrm{d}X(t) = (r + \alpha\lambda(t) - c(t))Y(t)\mathrm{d}t, \ Y(0) = \alpha X(0)$
- $\mathrm{d}Z(t) = (1-\alpha)\mathrm{d}X(t) = (r+\alpha\lambda(t)-c(t))Z(t)\mathrm{d}t, \ Z(0) = (1-\alpha)X(0)$

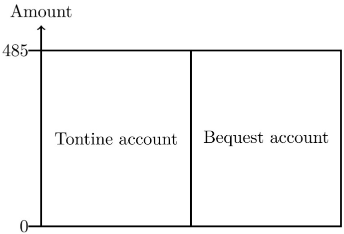
r : risk-free rate, $c(t)$: consumption rate

Illustration

(Continuing with our previous example)



(a) Before re-balancing.



(b) After consuming 50 units and re-balancing.

Bernhardt and Donnelly (2019,Fig.B.1.1)

- $\alpha = 0.5$
- consumption rate $= 6.25\%$
  $\rightarrow$consumption from each account $= 400 \times 6.25\% = 25$

Content

Utility

Consider the power utility function

$$U(x) = \frac{x^\gamma}{\gamma}, \quad \gamma \in (-\infty, 1)\backslash\{0\},\ x \geq 0$$

$1 - \gamma$: relative risk aversion level

Expected lifetime utility :

$$\mathbb{E}\left[\int_0^\tau e^{-\rho s}\frac{(c(s)X(s))^\gamma}{\gamma}\mathrm{d}s + be^{-\rho\tau}\frac{((1-\alpha)X(\tau))^\gamma}{\gamma}\right] =: \mathbb{E}\left[\mathcal{U}\right]$$

τ : lifetime random variable, ρ : time preference rate on consumption

b : strength of bequest motive

Content

Objective

- Given $\alpha \in [0,1]$ already fixed by retiree, we look for

$$\arg\max_{c(t)} \mathbb{E}\left[\int_0^\tau e^{-\rho s}\frac{(c(s)X(s))^\gamma}{\gamma}\mathrm{d}s + be^{-\rho\tau}\frac{((1-\alpha)X(\tau))^\gamma}{\gamma}\right]$$

- Constraint:
  - $\mathrm{d}X(t) = (r + \alpha\lambda(t) - c(t))X(t)\mathrm{d}t$

Solution

- By solving the Hamilton-Jacobi-Bellman equation
- Optimal consumption strategy:

$$c^\star(t) = (\gamma h(t))^{\frac{1}{\gamma-1}}$$

where $h(t)$ is the solution to

$0 = \partial_t h(t) + (1-\gamma)\gamma^{\frac{1}{\gamma-1}} h(t)^{\frac{\gamma}{\gamma-1}} + h(t)\psi(t) + \varphi(t)$
with $\psi(t) := \gamma(r + \alpha\lambda(t)) - \lambda(t) - \rho$
and $\varphi(t) := \frac{b}{\gamma}\lambda(t)(1-\alpha)^{\gamma}$.

Content

Objective

- We look for

$$\underset{\alpha,c}{\arg\max}\,\mathbb{E}\left[\int_0^{\tau} e^{-\rho s}\frac{(cX(s))^{\gamma}}{\gamma}\mathrm{d}s + be^{-\rho\tau}\frac{((1-\alpha)X(\tau))^{\gamma}}{\gamma}\right]$$

- Constraints:
  - $\mathrm{d}X(t) = [r + \alpha\lambda(t) - c(t)]X(t)\mathrm{d}t$
  - $\alpha \in [0,1]$
  - $c \geq 0$

Solution

- With some algebra, the expected lifetime utility can be written as

$$\mathbb{E}\left[\mathcal{U}\right] = \begin{cases} \frac{X(0)^\gamma}{\gamma}\left(b\frac{(1-\alpha)^\gamma}{1-\gamma\alpha}M_{A_\gamma}(-k) + c^\gamma\frac{1-M_{A_\gamma}(-k)}{k}\right), & \text{if } k \neq 0 \\ \frac{X(0)^\gamma}{\gamma}\left(b\frac{(1-\alpha)^\gamma}{1-\gamma\alpha} + c^\gamma\mathbb{E}\left[A_\gamma\right]\right), & \text{if } k = 0 \end{cases}$$

  where

  A_γ is random variable with tail distribution $\mathbb{P}(A_\gamma > t) = \mathbb{P}(\tau > t)^{1-\gamma\alpha}$,

  $k := \gamma(c-r)+\rho,\ M_{A_\gamma}(-k) = \mathbb{E}\left[e^{-kA_\gamma}\right].$

- No analytical solution for optimal α and c, to be solved numerically by evaluating the above expression

Solution

- With some algebra, the expected lifetime utility can be written as

$$\mathbb{E}\left[\mathcal{U}\right] = \begin{cases} \frac{X(0)^\gamma}{\gamma}\left(b\frac{(1-\alpha)^\gamma}{1-\gamma\alpha}M_{A_\gamma}(-k) + c^\gamma\frac{1-M_{A_\gamma}(-k)}{k}\right), & \text{if } k \neq 0 \\ \frac{X(0)^\gamma}{\gamma}\left(b\frac{(1-\alpha)^\gamma}{1-\gamma\alpha} + c^\gamma\mathbb{E}\left[A_\gamma\right]\right), & \text{if } k = 0 \end{cases}$$

  where

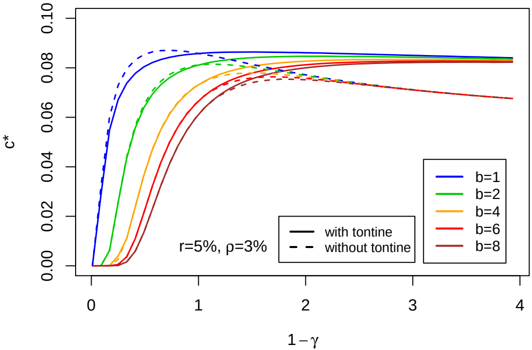
  A_γ is random variable with tail distribution $\mathbb{P}(A_\gamma > t) = \mathbb{P}(\tau > t)^{1-\gamma\alpha}$,

  $k := \gamma(c-r)+\rho,\ M_{A_\gamma}(-k) = \mathbb{E}\left[e^{-kA_\gamma}\right].$

- No analytical solution for optimal α and c, to be solved numerically by evaluating the above expression

Graphic results: $c^{\star}$

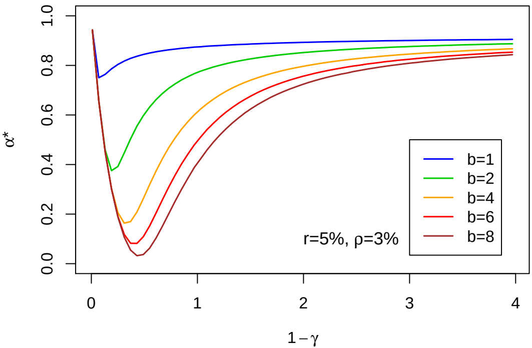
$\lambda(t) = 2.2 \cdot 10^{-4} + 2.7 \cdot 10^{-6} \cdot 1.124^{65+t}$



- Optimal consumption pattern with tontine is similar to that without tontine
- Higher consumption with tontine in most cases

Graphic results: $\alpha^\star$

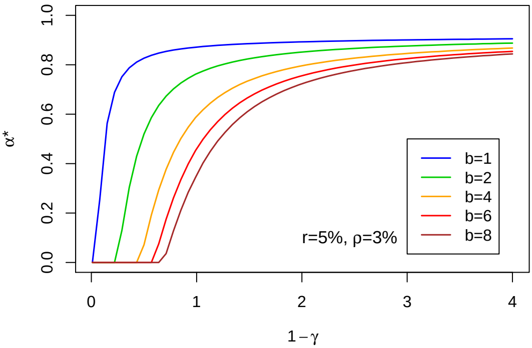
$\lambda(t) = 2.2 \cdot 10^{-4} + 2.7 \cdot 10^{-6} \cdot 1.124^{65+t}$



- Less risk averse: less investment in tontine
- Further less risk averse ($1-\gamma<0.5$): optimal investment in tontine increases sharply (why?)

Graphic results: $\alpha^\star$ if consider utility only until age 100

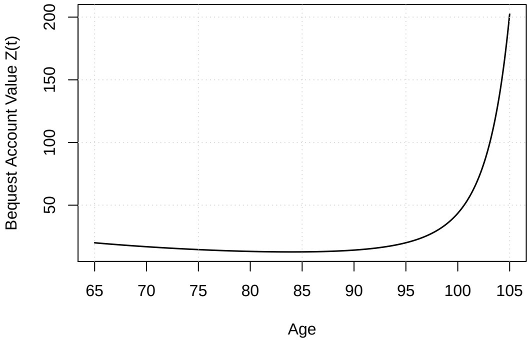
$\lambda(t) = 2.2 \cdot 10^{-4} + 2.7 \cdot 10^{-6} \cdot 1.124^{65+t}$



- Always decreasing investment in tontine when relative risk aversion decreases

Graphic results: projected bequest account value $Z(t)$

$\lambda(t) = 2.2 \cdot 10^{-4} + 2.7 \cdot 10^{-6} \cdot 1.124^{65+t}$, $X(0) = 100$, $\alpha = 0.8$, $c = 9\%$, $r = 5\%$.



- Bequest account value grows sharply if retiree survives until old age

Discussion

When risk aversion level is very small and $r > \rho$:

- Optimal consumption rate $c^\star$ decreases as relative risk aversion level decreases
- Optimal investment in tontine $\alpha^\star$ increases as relative risk aversion level decreases
- Explanation:
  - Rate of earning is greater than discounting rate on utility $\rightarrow$consuming later is better
  - "Gambling" on living very long time
  - By investing in tontine, retiree receives large amount of mortality credits at old age and hence benefits from the very high bequest account value

Discussion

When risk aversion level is very small and $r > \rho$:

- Optimal consumption rate $c^\star$ decreases as relative risk aversion level decreases
- Optimal investment in tontine $\alpha^\star$ increases as relative risk aversion level decreases
- Explanation:
  - Rate of earning is greater than discounting rate on utility $\rightarrow$consuming later is better
  - “Gambling” on living very long time
  - By investing in tontine, retiree receives large amount of mortality credits at old age and hence benefits from the very high bequest account value

Discussion

- Results are very sensitive for retirees with very low risk aversion and with bequest motive
  $\rightarrow$very precise choice of parameter values is needed
- Very risk averse retirees: stable $\alpha^\star$ and $c^\star$ regardless of strength of bequest motive
  $\rightarrow$bequest motive plays a small role in decision for tontine investment and consumption

Content

In the paper

- Two assets traded in financial market:
  - one risk-free asset earning risk-free rate r
  - one risky asset with mean return μ and volatility σ
- $\omega(t)$: portion of wealth invested in risky asset
- Dynamics of total account value:

$$\frac{\mathrm{d}X(t)}{X(t)} = [r + (\mu - r)\omega(t) + \alpha\lambda(t) - c(t)]\mathrm{d}t + \sigma\omega(t)\mathrm{d}W(t),$$

$X(0) > 0$ constant

$W(t)$: Brownian motion

In the paper

- Two assets traded in financial market:
  - one risk-free asset earning risk-free rate r
  - one risky asset with mean return μ and volatility σ
- $\omega(t)$: portion of wealth invested in risky asset
- Dynamics of total account value:

$$\frac{\mathrm{d}X(t)}{X(t)} = [r + (\mu - r)\omega(t) + \alpha\lambda(t) - c(t)]\mathrm{d}t + \sigma\omega(t)\mathrm{d}W(t),$$

$X(0) > 0$ constant

$W(t)$: Brownian motion

Results

- Also searched for optimal investment strategy $\omega^\star(t)$, which is found to be constant independent of α and $c(t)$
- Similar observations obtained for $\alpha^\star$ and $c^\star(t)$ as in risk-free case
- Not to be discussed in details in this seminar

Content

Contributions

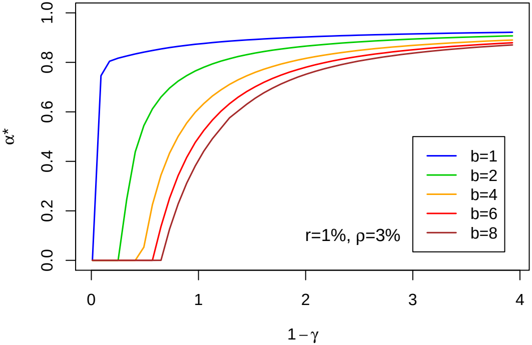
- New retirement product design which provides bequest to family after death
- Analysis on how bequest motive and risk aversion level influence investment and consumption strategies

Possible further research

- Finite pool?
- Stochastic mortality?
- Heterogeneous members?
- Comparison with other products?
- Bequest account for other products?

Graphic results: $\alpha^\star$ if $r < \rho$

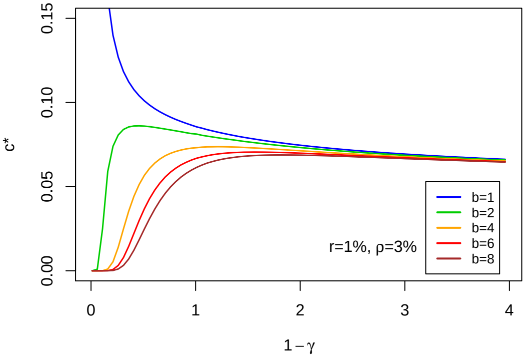
$\lambda(t) = 2.2 \cdot 10^{-4} + 2.7 \cdot 10^{-6} \cdot 1.124^{65+t}$



- Always decreasing investment in tontine when relative risk aversion decreases

Graphic results: $c^\star$ if $r < \rho$

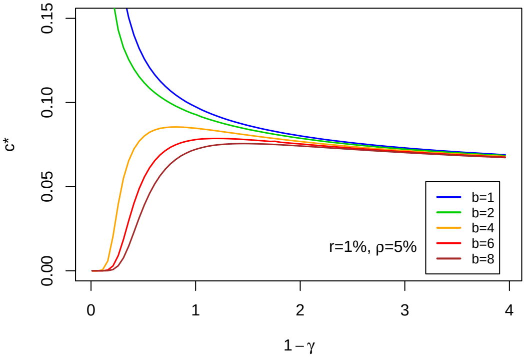
$\lambda(t) = 2.2 \cdot 10^{-4} + 2.7 \cdot 10^{-6} \cdot 1.124^{65+t}$



- Trend of $c^\star$ depends also on strength of bequest motive

Graphic results: $c^\star$ if $r < \rho$

$\lambda(t) = 2.2 \cdot 10^{-4} + 2.7 \cdot 10^{-6} \cdot 1.124^{65+t}$



- Trend of $c^\star$ depends also on strength of bequest motive