Seminar - Insurance Mathematics

Optimal retirement income tontines
Moshe A. Milevsky, Thomas S. Salisbury
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What is Tontine?

Proposition

Mathematical Proofs

Numerical Examples

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What is Tontine?

▶ A product to **pool longevity risk** of a group of people
  → retirement income product

▶ Sponsor, e.g. insurance company, is not exposed to any longevity risk

▶ Pool of participants bears the risk entirely
Example:

- 400 participants, each contributes €100, interest 4%
- yearly interest $400 \times 100 \times 4\% = €1600$ split among surviving participants
  - all survive: each receives €4
  - 40 survive: each receives €40
Brief History about Tontines

- Notable tontines in history:
  - first tontine: Lorenzo de Tonti, invented in 1650s, implemented in 1670 in Holland
  - 1693 in England by King William III
  - 1790 in U.S.: Hamilton’s Tontine Proposal

- Europe: lost popularity by 1850s

- U.S.: popular in late-19th century; banned since 1910

- traditional structure: fixed guaranteed payout rate
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A Tontine with properly-constructed payout can result in Expected Lifetime Utility comparable to that of a Life Annuity, which therefore is reasonable to exist in nowadays’ retirement insurance market.
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Approach

For both life-long Annuities and Tontines:

- Construct function for discounted expected lifetime utilities based on a payout function dependent on $t$
- Assume a constraint on the payout function
- Find optimal payout function
- Compare Utilities under optimal payout
Approach

For both life-long Annuities and Tontines:

- Construct function for **discounted expected lifetime utilities** based on a payout function dependent on $t$
- Assume a **constraint** on the payout function
- **Find optimal payout function** (*Euler-Lagrange theorem*)
- Compare Utilities under optimal payout
Approach

For both life-long Annuities and Tontines:

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\[
\begin{align*}
\text{maximize:} & \quad \int_{x_1}^{x_2} F(x, y, y') \, dx \\
\text{constraint:} & \quad \int_{x_1}^{x_2} G(x, y, y') \, dx = k \\
\Rightarrow & \quad \frac{\partial}{\partial y}(F + \lambda G) - \frac{d}{dt} \left[ \frac{\partial}{\partial y'}(F + \lambda G) \right] = 0
\end{align*}
\]
Lifetime Annuities
Lifetime Annuities

- Payout function $=: c(t)$
Lifetime Annuities

- Payout function $= c(t)$
- Discounted expected utilities

\[
E \left[ \int_0^\zeta e^{-rt} u(c(t)) \, dt \right] = \int_0^\infty e^{-rt} t p_x u(c(t)) \, dt
\]
Lifetime Annuities

- Payout function $= c(t)$

- Discounted expected utilities

$$= E \left[ \int_0^{\zeta} e^{-rt} u(c(t)) \, dt \right] = \int_0^{\infty} e^{-rt} t p_x u(c(t)) \, dt$$

- Payout constraint: $\int_0^{\infty} e^{-rt} t p_x c(t) \, dt = 1$
Lifetime Annuities

- Payout function \( =: c(t) \)

- Discounted expected utilities

\[
E \left[ \int_0^\zeta e^{-rt} u(c(t)) \, dt \right] = \int_0^\infty e^{-rt} t p_x u(c(t)) \, dt
\]

- Payout constraint:

\[
\int_0^\infty e^{-rt} t p_x c(t) \, dt = 1
\]

\[\text{Euler} \quad \text{Optimal payout } c(t) = \left[ \int_0^\infty e^{-rt} t p_x \, dt \right]^{-1} =: c_0\]
Tontines
Tontines

- Number of initial subscribers $=: n$
- Random number of live subscribers $=: N(t)$
- Given a live individual, assume number of other live subscribers $N(t) - 1 \sim Bin(n - 1, t\rho_x)$
Tontines

- Number of initial subscribers $=: n$
- Random number of live subscribers $=: N(t)$
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- Payout function $=: d(t)$
Tontines

- Number of initial subscribers $=: n$
- Random number of live subscribers $=: N(t)$
- Given a live individual, assume number of other live subscribers $N(t) - 1 \sim Bin(n - 1, t p_x)$

\[ \text{Payout function } =: d(t) \]

\[ \text{Discounted expected utilities} \]

\[ = E \left[ \int_0^\zeta e^{-rt} u \left( \frac{nd(t)}{N(t)} \right) dt \right] \]

\[ = \int_0^\infty e^{-rt} t p_x \sum_{k=0}^{n-1} \binom{n-1}{k} t p_x^k (1 - t p_x)^{n-1-k} u \left( \frac{nd(t)}{k + 1} \right) dt \]
Payout constraint: \( \int_0^\infty e^{-rt} d(t) dt = 1 \)
Payout constraint: \[ \int_0^\infty e^{-rt} d(t) dt = 1 \]

\[ \Rightarrow \]

Optimal payout \( d(t) = D_u(t p_x) \),

which \( D_u(p) \) satisfies

\[ p \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1 - p)^{n-1-k} \frac{n}{k+1} u' \left( \frac{n D_u(p)}{k+1} \right) = \lambda \]

and \( \lambda \) is chosen such that the payout constraint is satisfied.
In case of **Constant Relative Risk Aversion (CRRA)**:

\[
u(c) = \begin{cases} 
  c^{1-\gamma}/(1 - \gamma) & \text{if } \gamma \neq 1 \\
  \log c & \text{if } \gamma = 1 
\end{cases}
\]
In case of **Constant Relative Risk Aversion (CRRA)**:

\[
u(c) = \begin{cases} 
  c^{1-\gamma}/(1 - \gamma) & \text{if } \gamma \neq 1 \\
  \log c & \text{if } \gamma = 1
\end{cases}
\]

**Optimal payout:**

\[
D_{n,\gamma}^{OT}(t\rho_x) = D_{n,\gamma}^{OT}(1) \beta_{n,\gamma}(t\rho_x)^{1/\gamma}
\]

where

\[
\beta_{n,\gamma}(p) = p \cdot E\left[\left(\frac{n}{N(p)}\right)^{1-\gamma}\right], \quad N(p) - 1 \sim Bin(n - 1, p)
\]

\[
D_{n,\gamma}^{OT}(1) = \left[\int_0^\infty e^{-rt} \beta_{n,\gamma}(t\rho_x)^{1/\gamma} dt\right]^{-1}
\]
Compare Expected Utilities
Compare Expected Utilities

Annuity:

\[ U^A_{\gamma} = \frac{1}{1 - \gamma} \left( \int_0^\infty e^{-rt} t p_x dt \right)^\gamma \]

Tontine:

\[ U^{OT}_{n,\gamma} = \frac{1}{1 - \gamma} \left( \int_0^\infty e^{-rt} \beta_{n,\gamma}(t p_x)^{1/\gamma} dt \right)^\gamma \]
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\[ U^{OT}_{n,\gamma} = \frac{1}{1 - \gamma} \left( \int_0^\infty e^{-rt} \beta_{n,\gamma}(tp_x)^{1/\gamma} dt \right)^\gamma \]

\[ \beta_{n,\gamma}(p) \begin{cases} < p^\gamma & \text{if } 0 < \gamma < 1 \\ > p^\gamma & \text{if } 1 < \gamma \end{cases} \]
Compare Expected Utilities

Annuity:

\[ U^{A}_{\gamma} = \frac{1}{1 - \gamma} \left( \int_{0}^{\infty} e^{-rt} t p_x dt \right)^\gamma \]

Tontine:

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\[ \beta_{n,\gamma}(p) \begin{cases} < p^\gamma & \text{if } 0 < \gamma < 1 \\ > p^\gamma & \text{if } 1 < \gamma \end{cases} \]

⇒ A Tontine always has expected utility smaller than that of an Annuity.
What if there are costs for an insurance company to finance the annuity?

E.g. Capital reserves, risk management costs...
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→ A fraction $\delta$ of initial deposits is deducted as funding.
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E.g. Capital reserves, risk management costs...

→ A fraction $\delta$ of initial deposits is deducted as funding.

Utility of Loaded Annuity:

$$U^A_{\gamma} = \frac{(1 - \delta)^{1-\gamma}}{1 - \gamma} \left( \int_0^\infty e^{-rt} t^\gamma p_x dt \right)^\gamma$$
What if there are costs for an insurance company to finance the annuity?

E.g. Capital reserves, risk management costs...

→ A fraction $\delta$ of initial deposits is deducted as funding.

Utility of Loaded Annuity:

$$U_A^\gamma = \frac{(1 - \delta)^{1-\gamma}}{1 - \gamma} \left( \int_0^\infty e^{-rt} tp_x dt \right)^\gamma$$

Tontine: $U_{n,\gamma}^{OT} = \frac{1}{1 - \gamma} \left( \int_0^\infty e^{-rt} \beta_{n,\gamma}(tp_x)^{1/\gamma} dt \right)^\gamma$
Conclusion

If costs are incurred to finance an annuity, a tontine may offer a higher expected lifetime utility than an annuity.
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Cashflow Ranges of Flatrate Tontine

Range of Flat 4% Tontine Payout Purchased at 65: Gompertz Mortality
10th vs. 90th percentile: n = 400 (m=88.721, b=10)
Cashflow Ranges of Flatrate Tontine

- Low in early years, highly variable in final years.
Optimal Tontine Payout Function

Optimal Tontine Payout Function: Gompertz Mortality
Age = 65; n = 250; Risk Free Rate = 4%, m = 88.721, b = 10
Optimal Tontine Payout Function

- Guaranteed rate is higher in early years, then declines over time.
Optimal Tontine Payout Function

- Guaranteed rate is higher in early years, then declines over time.
- Difference in payout for different $\gamma$’s is barely noticeable.
Optimal Tontine Payout Function

Optimal tontine payout function: pool of size $n = 25$

<table>
<thead>
<tr>
<th>LoRA ($\gamma$)</th>
<th>Payout age 65</th>
<th>Payout age 80</th>
<th>Payout age 95</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>7.565%</td>
<td>5.446%</td>
<td>1.200%</td>
</tr>
<tr>
<td>1.0</td>
<td>7.520%</td>
<td>5.435%</td>
<td>1.268%</td>
</tr>
<tr>
<td>1.5</td>
<td>7.482%</td>
<td>5.428%</td>
<td>1.324%</td>
</tr>
<tr>
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<td>5.423%</td>
<td>1.374%</td>
</tr>
<tr>
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<td>7.324%</td>
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<td>1.541%</td>
</tr>
<tr>
<td>9.0</td>
<td>7.081%</td>
<td>5.394%</td>
<td>1.847%</td>
</tr>
</tbody>
</table>

**Survival**

- $0p_{65} = 100\%$
- $15p_{65} = 72.2\%$
- $30p_{65} = 16.8\%$

Notes: Assumes $r = 4\%$ and Gompertz Mortality ($m = 88.72, b = 10$).
### Optimal Tontine Payout Function

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Notes: Assumes $r = 4\%$ and Gompertz Mortality ($m = 88.72, b = 10$).

- Optimal tontine payout rates are insensitive to Risk Aversion Level $\gamma$, even if pool size is small.
Cashflow Ranges of Optimal Tontine

Range of Optimal Tontine Payout at 4% Interest: Gompertz Mortality
10th vs. 90th percentile: n = 400 (m=88.721, b=10)
Cashflow Ranges of Optimal Tontine

- Higher in initial years, relatively stable in final years.
Cashflow Ranges of Flatrate vs. Optimal Tontine

Flatrate:

Optimal:
Payout Functions of Optimal Tontine vs. Annuity

Optimal Tontine Payout Function vs. Annuity with Loading
Age = 65; n = 400; Risk Free Rate = 4%, m = 88.721, b = 10

- any gamma
- Fair Annuity Payout
- Perpetuity
- Annuity with Loading
Payout Functions of Optimal Tontine vs. Annuity

- A loading to an annuity drives its payout down, hence utility of an annuity may be lower than that of an optimal tontine.
Utility Indifference Loadings

If the risk loading $\delta$ the annuity charges up front is higher than these amounts, the tontine is preferred.

<table>
<thead>
<tr>
<th>LoRA $\gamma$</th>
<th>$n = 20$</th>
<th>$n = 100$</th>
<th>$n = 500$</th>
<th>$n = 1000$</th>
<th>$n = 5000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>129.8 b.p.</td>
<td>27.4 b.p.</td>
<td>5.74 b.p.</td>
<td>2.92 b.p.</td>
<td>0.60 b.p.</td>
</tr>
<tr>
<td>9.0</td>
<td>753.6 b.p.</td>
<td>199.8 b.p.</td>
<td>45.9 b.p.</td>
<td>23.8 b.p.</td>
<td>5.09 b.p.</td>
</tr>
</tbody>
</table>

Assumes age $x = 60$, $r = 3\%$ and Gompertz mortality ($m = 87.25$, $b = 9.5$).
Natural Tontine
Natural Tontine

- Payout:

\[ d(t) = D_N(t \rho_x) \propto t \rho_x \]
Natural Tontine

- Payout:

\[ d(t) = D_N(t \rho_x) \propto t \rho_x \]

Constraints:

\[
\int_0^\infty e^{-rt} t \rho_x c(t) dt = 1 \\
\int_0^\infty e^{-rt} d(t) dt = 1
\]
Natural Tontine

Payout:

\[ d(t) = D_N(t\rho_x) \propto t\rho_x \]
\[ = t\rho_x c_0 \]

Constraints:

\[ \int_0^\infty e^{-rt} t\rho_x c(t) dt = 1 \]
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Natural Tontine

- Payout:
  \[ d(t) = D_N(t\rho_x) \propto t\rho_x \]
  \[ = t\rho_x c_0 \]

- Optimal for risk aversion
  \[ \gamma = 1 \]

Constraints:
\[
\begin{align*}
\int_0^\infty e^{-rt}t\rho_x c(t)dt &= 1 \\
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\end{align*}
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Natural Tontine

- **Payout:**
  \[
  d(t) = D_N(t\rho_x) \propto t\rho_x \\
  = t\rho_x c_0
  \]

- **Optimal for risk aversion**
  \[\gamma = 1\]

Constraints:
\[
\int_0^\infty e^{-rt} t\rho_x c(t) dt = 1 \\
\int_0^\infty e^{-rt} d(t) dt = 1
\]

Optimal tontine:
\[
D^{OT}_{n,\gamma}(p) = D^{OT}_{n,\gamma}(1) \beta_{n,\gamma}(p)^{1/\gamma}
\]
Certainty Equivalent factors associated with Natural Tontine

How much to be invested in the Natural Tontine in order to match €1 invested in a tontine optimized for the individual’s own risk aversion.

<table>
<thead>
<tr>
<th>Natural vs. optimal tontine</th>
</tr>
</thead>
<tbody>
<tr>
<td>Certainty equivalent for $n = 100$</td>
</tr>
<tr>
<td>Age $x$</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>40</td>
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</tr>
<tr>
<td>60</td>
</tr>
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</tr>
<tr>
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$r = 3\%$ and Gompertz $m = 87.25, b = 9.5$. 
Certainty Equivalent factors associated with Natural Tontine

- How much to be invested in the Natural Tontine in order to match €1 invested in a tontine optimized for the individual’s own risk aversion.

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</tr>
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<td>1.000026</td>
<td>1</td>
<td>1.000753</td>
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<td>1.001674</td>
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<tr>
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<td>1</td>
<td>1.003388</td>
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<tr>
<td>70</td>
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<td>1</td>
<td>1.003451</td>
</tr>
<tr>
<td>80</td>
<td>1.000225</td>
<td>1</td>
<td>1.009877</td>
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\( r = 3\% \) and Gompertz \( m = 87.25, b = 9.5 \).

- Welfare loss for an individual with \( \gamma \neq 1 \) to buy a Natural Tontine is minimal.
Certainty Equivalent factors associated with Natural Tontine

How much to be invested in the Natural Tontine in order to match €1 invested in a tontine optimized for the individual’s own risk aversion.

Natural vs. optimal tontine

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$r = 3\%$ and Gompertz $m = 87.25, b = 9.5$.

- Welfare loss for an individual with $\gamma \neq 1$ to buy a Natural Tontine is minimal.
- Basis for designing tontine products.
Summary

- The Optimal Tontine offers a more desirable payout structure than the historical Flatrate Tontine.
- The Optimal Tontine is possible to offer a higher expected lifetime utility than a Loaded Annuity.
- The Natural Tontine is a reasonable structure for designing tontine products in practice.
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Problems in modern insurance industry
- Longevity risks capacity
- Poor experience in managing risk of long-dated fixed guarantees

Solvency II
- Insurance companies have to hold more capital against risks
- Higher prices for products with long-term guarantees

Annuity puzzle
Tontines could be a solution.

Other recent research in tontine-like structures:
Piggott et al. (2005), Valdez et al. (2006), Stamos (2008), Richter and Weber (2011), Donnelly et al. (2013), Qiao and Sherris (2013) ...
Outlook

Problems in modern insurance industry

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Outlook

- Problems in modern insurance industry
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Outlook

▶ Problems in modern insurance industry
  ▶ longevity risks capacity
  ▶ poor experience in managing risk of long-dated fixed guarantees

▶ Solvency II
  ▶ insurance companies have to hold more capital against risks
  ▶ higher prices for products with long-term guarantees

▶ Annuity puzzle

**Tontines could be a solution.**

Other recent research in tontine-like structures:
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Limitations
Limitations and Future Works

- Credit risk: default of sponsor
- Stochastic mortality: mortality rates change over time
  - tontine payouts more uncertain
  - higher capital charges added to annuity
- Asymmetric mortality: individuals have more information about their own life
Limitations and Future Works

- Credit risk: default of sponsor

Stochastic mortality: mortality rates change over time, leading to more uncertain tontine payouts.

Higher capital charges are added to annuities.

Asymmetric mortality: individuals have more information about their own life.
Limitations and Future Works

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