Seminar - Insurance Mathematics

Optimal retirement income tontines

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Content

Content

What is Tontine?

- A product to **pool longevity risk** of a group of people $\rightarrow$ retirement income product
- Sponsor, e.g. insurance company, is not exposed to any longevity risk
- Pool of participants bears the risk entirely

Example:

- 400 participants, each contributes €100, interest 4%
- yearly interest $400 \times 100 \times 4\% =$ €1600 split among surviving participants
  - all survive: each receives €4
  - 40 survive: each receives €40

Brief History about Tontines

- Notable tontines in history:
  - first tontine: Lorenzo de Tonti, invented in 1650s, implemented in 1670 in Holland
  - 1693 in England by King William III
  - 1790 in U.S.: Hamilton's Tontine Proposal
- Europe: lost popularity by 1850s
- U.S.: popular in late-19th century; banned since 1910
- traditional structure: fixed guaranteed payout rate

Content

Proposition

*A **Tontine with properly-constructed payout** can result in **Expected Lifetime Utility** comparable to that of a Life Annuity, which therefore is reasonable to exist in nowadays' retirement insurance market.*

Content

Approach

For both life-long Annuities and Tontines:

- Construct function for **discounted expected lifetime utilities** based on a payout function dependent on t
- Assume a **constraint** on the payout function
- **Find optimal payout function**

➔ Compare Utilities under optimal payout

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maximize: $\int_{x_1}^{x_2} F(x, y, y')dx$

constraint: $\int_{x_1}^{x_2} G(x, y, y')dx = k$

$\Rightarrow \frac{\partial}{\partial y}(F + \lambda G) - \frac{d}{dt}\left[\frac{\partial}{\partial y'}(F + \lambda G)\right] = 0$

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$\overset{\text{Euler}}{\Longrightarrow}$ Optimal payout $c(t) = \left[\int_0^{\infty} e^{-rt} {}_tp_x dt\right]^{-1} =: c_0$

Tontines

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- Number of initial subscribers $=: n$
- Random number of live subscribers $=: N(t)$
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- ▶ Discounted expected utilities

$$= E\left[\int_0^{\zeta} e^{-rt} u\left(\frac{nd(t)}{N(t)}\right) dt\right]$$

$$= \int_0^{\infty} e^{-rt}{}_tp_x \sum_{k=0}^{n-1} \binom{n-1}{k} {}_tp_x^k(1 - {}_tp_x)^{n-1-k} u\left(\frac{nd(t)}{k+1}\right) dt$$

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$\xRightarrow{\text{Euler}}$ Optimal payout $d(t) = D_u({}_t p_x)$,

which $D_u(p)$ satisfies

$$p \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \frac{n}{k+1} u' \left(\frac{n D_u(p)}{k+1} \right) = \lambda$$

and λ is chosen such that the payout constraint is satisfied.

In case of Constant Relative Risk Aversion (CRRA):

$$u(c) = \begin{cases} c^{1-\gamma}/(1-\gamma) & \text{if } \gamma \neq 1 \\ \log c & \text{if } \gamma = 1 \end{cases}$$

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Optimal payout: $D^{OT}_{n,\gamma}({}_tp_x) = D^{OT}_{n,\gamma}(1)\beta_{n,\gamma}({}_tp_x)^{1/\gamma}$

where $\beta_{n,\gamma}(p) = p \cdot E\left[\left(\frac{n}{N(p)}\right)^{1-\gamma}\right], N(p) - 1 \sim Bin(n-1, p)$

$$D^{OT}_{n,\gamma}(1) = \left[\int_0^\infty e^{-rt}\beta_{n,\gamma}({}_tp_x)^{1/\gamma}dt\right]^{-1}$$

Compare Expected Utilities

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Annuity:

$$U_\gamma^A = \frac{1}{1-\gamma}\left(\int_0^\infty e^{-rt}\,{}_tp_x\,dt\right)^\gamma$$

Tontine:

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$\Rightarrow$ A Tontine always has expected utility smaller than that of an Annuity.

*What if there are **costs** for an insurance company to finance the annuity?*

E.g. Capital reserves, risk management costs...

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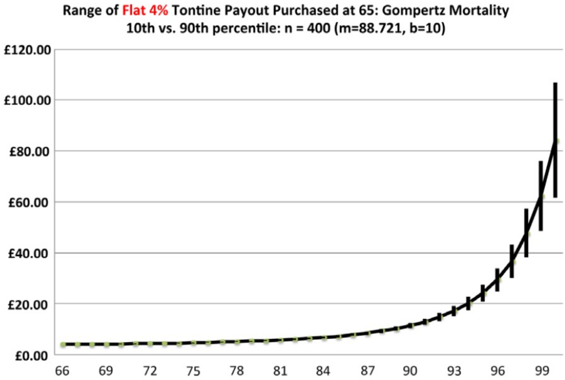
Tontine: $U_{n,\gamma}^{OT} = \frac{1}{1-\gamma} \left(\int_0^\infty e^{-rt} \beta_{n,\gamma} ({}_tp_x)^{1/\gamma} dt \right)^\gamma$

Conclusion

If costs are incurred to finance an annuity, a tontine may offer a higher expected lifetime utility than an annuity.

Content

Cashflow Ranges of Flatrate Tontine



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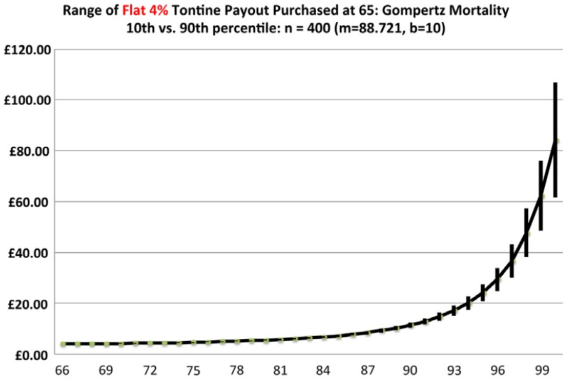
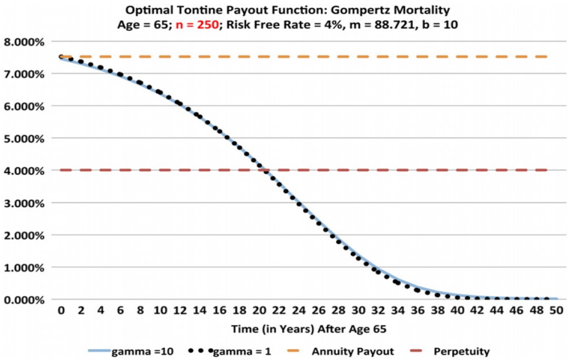


- Low in early years, highly variable in final years.

Optimal Tontine Payout Function



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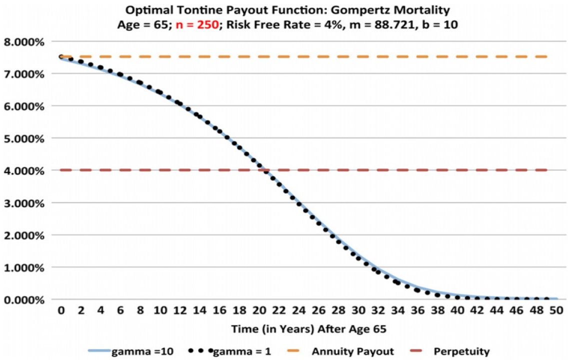

- Guaranteed rate is higher in early years, then declines over time.

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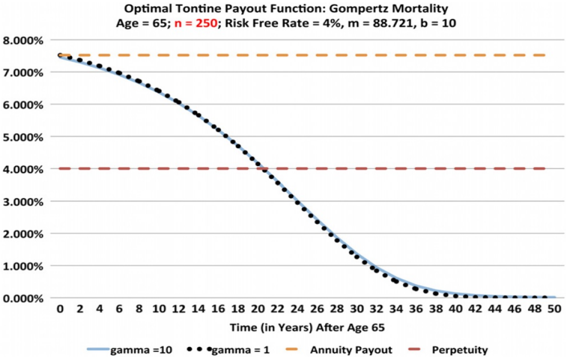

- Guaranteed rate is higher in early years, then declines over time.
- Difference in payout for different γ's is barely noticeable.

Optimal Tontine Payout Function

Optimal tontine payout function: pool of size $n = 25$			
LoRA (γ)	Payout age 65	Payout age 80	Payout age 95
0.5	7.565%	5.446%	1.200%
1.0	7.520%	5.435%	1.268%
1.5	7.482%	5.428%	1.324%
2.0	7.447%	5.423%	1.374%
4.0	7.324%	5.410%	1.541%
9.0	7.081%	5.394%	1.847%
Survival	${}_0p_{65} = 100\%$	${}_{15}p_{65} = 72.2\%$	${}_{30}p_{65} = 16.8\%$

Notes: Assumes $r = 4\%$ and Gompertz Mortality ($m = 88.72$, $b = 10$).

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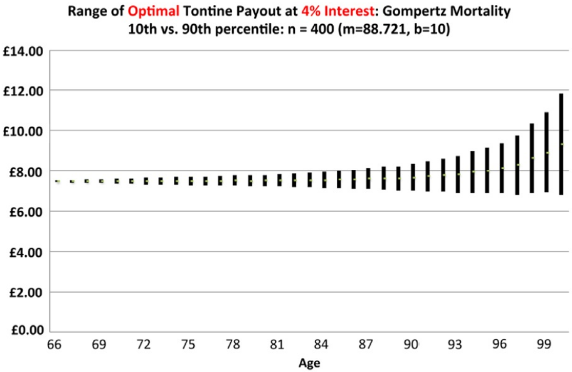
- Optimal tontine payout rates are insensitive to Risk Aversion Level γ, even if pool size is small.

Cashflow Ranges of Optimal Tontine



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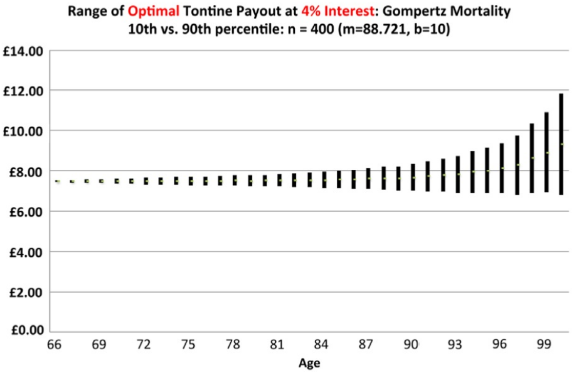
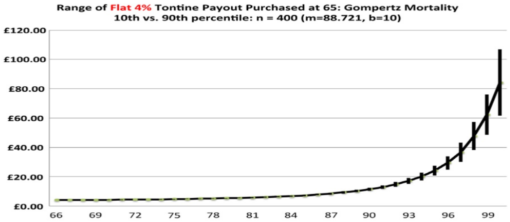


- Higher in initial years, relatively stable in final years.

Cashflow Ranges of Flatrate vs. Optimal Tontine

Flatrate:



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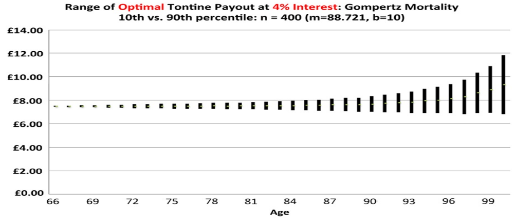
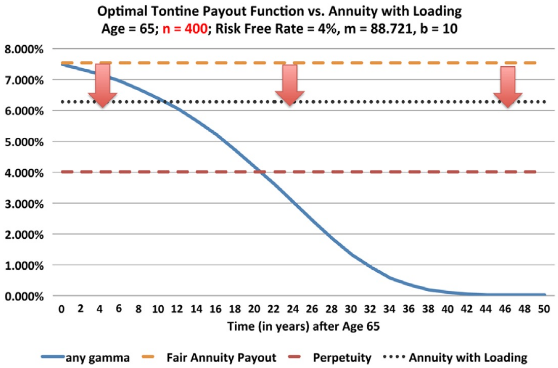


Payout Functions of Optimal Tontine vs. Annuity



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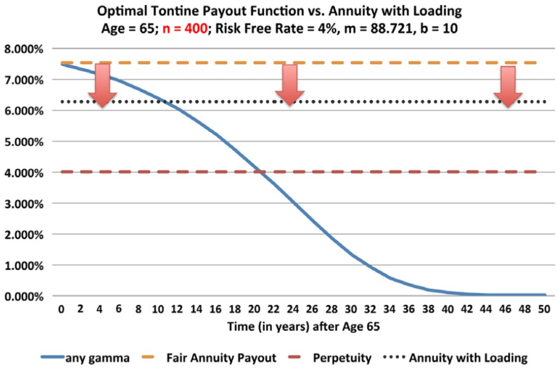

- A loading to an annuity drives its payout down, hence utility of an annuity may be lower than that of an optimal tontine.

Utility Indifference Loadings

If the risk loading δ the annuity charges up front is higher than these amounts, the tontine is preferred.

The highest annuity loading δ you are willing to pay

If a tontine pool of size n is available

LoRA γ	$n = 20$	$n = 100$	$n = 500$	$n = 1000$	$n = 5000$
0.5	72.6 b.p.	14.5 b.p.	2.97 b.p.	1.50 b.p.	0.30 b.p.
1.0	129.8 b.p.	27.4 b.p.	5.74 b.p.	2.92 b.p.	0.60 b.p.
1.5	182.4 b.p.	39.8 b.p.	8.45 b.p.	4.31 b.p.	0.89 b.p.
2.0	231.7 b.p.	51.8 b.p.	11.1 b.p.	5.68 b.p.	1.18 b.p.
3.0	323.1 b.p.	75.1 b.p.	16.3 b.p.	8.38 b.p.	1.75 b.p.
9.0	753.6 b.p.	199.8 b.p.	45.9 b.p.	23.8 b.p.	5.09 b.p.

Assumes age $x = 60$, $r = 3\%$ and Gompertz mortality ($m = 87.25$, $b = 9.5$).

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Certainty Equivalent factors associated with Natural Tontine

- How much to be invested in the Natural Tontine in order to match €1 invested in a tontine optimized for the individual's own risk aversion.

Natural vs. optimal tontine

Certainty equivalent for $n = 100$			
Age x	$\gamma = 0.5$	$\gamma = 1$	$\gamma = 2$
30	1.000018	1	1.000215
40	1.000026	1	1.000753
50	1.000041	1	1.001674
60	1.000067	1	1.003388
70	1.000118	1	1.003451
80	1.000225	1	1.009877

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- Welfare loss for an individual with $\gamma \neq 1$ to buy a Natural Tontine is minimal.
- Basis for designing tontine products.

Summary

- The Optimal Tontine offers a more desirable payout structure than the historical Flatrate Tontine.
- The Optimal Tontine is possible to offer a higher expected lifetime utility than a Loaded Annuity.
- The Natural Tontine is a reasonable structure for designing tontine products in practice.

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Tontines could be a solution.

Other recent research in tontine-like structures:

Piggott et al.(2005), Valdez et al.(2006), Stamos(2008), Richter and Weber(2011), Donnelly et al.(2013), Qiao and Sherris(2013)...

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Limitations and Future Works

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- Stochastic mortality : mortality rates change over time
  - tontine payouts more uncertain
  - higher capital charges added to annuity
- Asymmetric mortality : individuals have more information about his/her own life