Ulm University
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Towards Practical Annuity Overlay Fund

Master Thesis
in Finance
(major: Actuarial Science)

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Abstract

The annuity overlay fund by Donnelly et al. [Donnelly, C., Guillén, M., Nielsen, J.P., 2014. Bringing cost transparency to the life annuity market. Insur Math Econ 56, 14-27] pools mortality risks of a group of individuals. Comparing to traditional annuities, the fund has the merits of being cost-transparent, granting members the authority on investment decisions, as well as allowing free entry and exit before death. The great flexibility of the annuity overlay fund enables itself to be adapted for applications in various aspects.

We explore the fund in three directions:

- a discrete-time model for the annuity overlay fund is developed, which is helpful for practical implementation of the fund;
- the risk and return features of the fund are examined and a framework to analyse the relative attractiveness of the fund between members with heterogeneous wealth-mortality profiles is proposed; and
- a specific method to operate the fund is proposed for application in the retirement aspect, with which a stream of benefit payment that is constant in expectation over time can be provided to fund members until death.

The insights brought out from the thesis serve to bring readers a more thorough picture of the annuity overlay fund and provoke further investigation on its usefulness in practice. Our discussions and results can also be seen as a confirmation of the practicality of the fund.

Contents

Chapter 1

Introduction

Annuity overlay fund, recently proposed by Donnelly et al. (2014), is an investment product allowing to pool mortality risk. In this thesis, we follow up the work of Donnelly et al. (2014) by developing a discrete-time model, conducting a risk-return analysis, and proposing an application in the field of retirement insurance.

1.1 Annuity Overlay Fund

Annuities have long been used globally as a means of retirement insurance. At a high level, an annuitant pays a single premium at inception, in exchange for a fixed payment to be received regularly until death. The uncertainty in the death time of the annuitant, termed as the mortality risk, is transferred to the insurer through buying the annuity. Regardless of until when the annuitant survives, she is protected financially by the fixed benefit payments.

The importance of annuitization in retirement planning has been well recognized among the literature since the work by Yaari (1965), which has shown that, individuals who have no bequest motive should annuitize all wealth so as to maximize the lifetime expected utility, assuming that the annuity market is actuarially fair. Nevertheless, throughout the decades, demand for annuity has been observed to be significantly lower than the level predicted by theories, a phenomenon termed as the annuity puzzle. Among others, the fact that annuities are cost-intransparent has been suggested to be one of the reasons of the puzzle (Blake (1999); Stewart (2007)). See Section 1.5 for a further discussion on the cost-intransparency issue with annuities. In view of this issue, a new product structure, called annuity overlay fund, has been recently proposed by Donnelly et al. (2014) and is reviewed below.

The annuity overlay fund, being similar to an annuity, pools mortality risks of a group of individuals. The fund operates as follows: at inception, a group of surviving participants joins the fund by investing some amount of wealth. This wealth is to be accumulated with market return through investing in the financial market until some fixed terminal time. Then, at the terminal time, the wealth from participants who have perished by this time will be shared among all members in the fund (including those who have perished), with each member receiving an amount computed based on both her probability of death in the whole period and her accumulated wealth at the terminal time. The annuity overlay fund offers a number of advantages over an annuity.

- **Cost-transparency.** Under the mechanism of the annuity overlay fund, mortality risk and investment risk are separated. This allows various costs to be ascribed to their own source. For example, customers can identify investment fees, administrative costs and other charges independently. A transparent disclosure of charges can be achieved.
- **Freedom of investment decision.** Participants of the annuity overlay fund can freely decide their own investment strategy in the financial market, and the market risk is born by themselves. This is in contrast with an annuity, with which investment decisions are determined by the insurer and customers have no authority over it.
- **Freedom of entry and exit.** Unlike with an annuity, where customers must remain in the contract until death, or otherwise pay a substantial financial penalty upon exit, individuals can enter and exit the annuity overlay fund before death without penalty. In the limiting case, as in the work by Donnelly et al. (2014), where the mathematical model is built on an instantaneous time basis, individuals can freely enter and exit the fund at any point of time. In this thesis we consider a discrete-time model, under which individuals can freely enter and exit the fund at the beginning of each period specified by the fund provider.
- **Investment framing.** The annuity overlay fund offers participants tangible financial gain from pooling mortality risks. Participants are able to assess the fund by simply looking at the yield obtained, same as when making other investment decisions. This is different from annuities, the value of which is appraised under a consumption framing. It has been documented that, when evaluating financial benefits from pooling mortality risks, individuals tend

to adopt an investment framing, for example, by relating an annuity to the risk and return features or a "payback period" (Hu and Scott (2007); Brown et al. (2008)). In this regard, the annuity overlay fund is likely to appear more attractive than annuities to consumers.

Along with the above merits, since the annuity overlay fund is merely pooling and redistributing wealth of participants, there is no guarantee involved and hence no risk capital requirement is to be fulfilled. Consequently the annuity overlay fund can be offered to customers at a low cost.

It has been emphasized in the work by Donnelly et al. (2014) that, since the annuity overlay fund does not guarantee participants an income stream until death, i.e. not protecting participants against mortality risk, being different from an annuity, the fund should not be treated as an insurance product but rather an investment product. The purpose of the fund is to offer consumers an alternative way of pooling mortality risks, with which costs can be transparently disclosed and traced to respective sources, allowing consumers to make informed financial decisions and purchase what really suits their need.

Donnelly et al. (2014) provide theoretical formulas for the expectation and variance of the fund's payout to a fund member, which is dependent on the member's wealth and probability of death in the concerned period (a.k.a. her wealth-mortality profile), yet there are limited elaborations on them. While different members receive some different payouts, it is not clear, how these are to be valued from the perspective of the fund members. In this regard, we raise our first question:

> *Is the fund equally attractive to members with different wealth-mortality profiles, or would any particular group find the fund more favourable than the others?*

Meanwhile, due to improving human life expectancy, insurance companies are facing increasing pressure for providing retirement products including annuities. As retirees generally live longer, retirement benefits have to be provided for a longer period on average. The tightening regulations, such as the introduction of Solvency II in Europe, together with the low interest rate environment in recent years have added further difficulty to insurers in meeting the risk capital requirements for providing retirement policies. One direct consequence of these issues is an increase in annuity price to cover the growing costs. Eventually, buyers of retirement insurance suffer from paying a higher insurance premium.

In attempt to mitigate the problem, various alternative product designs which pool mortality risks across a group of participants have been reviewed and studied in recent years. For example, tontine and pooled annuity funds have been much investigated. See Section 1.5 below for a brief review on these structures. Together with the annuity overlay fund, these structures all work fundamentally with the same central idea: pooling the wealth of a group of individuals and redistributing the wealth of the deceased ones among the group according to certain principle. For all, the distinction lies in how the wealth from the deceased is distributed.

In recent research, both the tontine and the pooled annuity fund have been suggested as a means of supporting retirement consumption. Instead of relying on annuities, which require the insurer to bear all longevity risks of retirees, these structures are proposed, as they can pool the mortality risks of a group of retirees in a way that allows the retirees to bear part of the risks themselves. The part of risk born by the retirees is reflected through the uncertainty in their actual retirement benefit amount to be received. Through these innovations, it is hoped that a better splitting of risk between the insurer and the retirees can be attained, which is beneficial to both parties in the sense that, insurer bears less risk and hence faces less pressure from the risk capital requirement, whereas retirees are charged a lower premium in exchange for bearing part of the risk themselves.

In view of the similar concept behind all these structures, we are motivated to look at the possibility of applying the annuity overlay fund to support retirement consumption. We ask the second question:

> *Could the annuity overlay fund be managed in some way, such that it could also serve to provide retirement benefits to fund members?*

1.2 Our Contributions

This thesis serves to address the two raised questions above and contributes in the following aspects:

1. Based on the continuous-time model of the annuity overlay fund proposed in Donnelly et al. (2014), we establish a discrete-time model of the fund. The discrete-time model is employed, since it is often preferred by insurance companies for business execution in practice.

2. The risk and return features of the annuity overlay fund are investigated, and an analysis based on these features is proposed to examine the relative attractiveness of the fund between members with heterogeneous wealth-mortality profiles. See Section 1.3 for an overview.

3. A novel way of operating the annuity overlay fund is proposed, with which the fund acts as an instrument to provide retirement benefits to fund members. The resulting stream of retirement benefits is constant in expectation over time, and the insurance company bears theoretically zero risk for providing such payments. See Section 1.4 for an overview.

1.3 An Analysis based on Risk and Return

The annuity overlay fund offers fund members two sources of gain (or loss). First, the initial wealth of a member is accumulated with return from the financial market until some fixed terminal time, according to some investment strategy freely chosen by the member. Then, at the terminal time, both the accumulated wealth of the member and her survival status until then will be used to determine her second gain from the fund, the gain from pooling mortality risks, termed by the original authors as the actuarial gain.

In our upcoming analysis, we view the expectation of actuarial gain upon survival at the terminal time as "return" and the variance of which as "risk", both quantities conditional on that the market return is known. By referencing to these quantities, we examine the attractiveness of the annuity overlay fund for participants with different wealth-mortality profiles.

Our setting for "return" and "risk" as such is based on two considerations:

1. <u>Randomness in market return not considered</u>

   Our focus is on the pooling of mortality risks, that is, how the actuarial gain for the fund members are ascertained based on their wealth-mortality profiles at the terminal time. Although market return plays a role in determining the terminal wealth, it is not the center of our attention. For simplicity, our analysis is done conditional on that the market return for all fund members is already known. [1]

2. No bequest motive

   We assume that bequest motive has been taken care of by all individuals before they commit any wealth into the fund, so that there is no motive to obtain bequest from the fund. Therefore, the actuarial gain to be received upon death by the terminal time is left out, only the gain upon survival is considered in our analysis. This point is worth some elaboration. As shown in the work by Donnelly et al. (2014) and also in our revisit on the fund later, unconditional on the survival status at the terminal time, a fund member earns an expected actuarial gain of zero. Yet, the variance of the actuarial gain, which is random, is obviously greater than zero. Therefore, from the perspective of risk and return, it would not be rational for any risk-averse individual to invest into the fund if one takes both the cases of survival and death into consideration. Meanwhile, as we will also see, the actuarial gain upon death is most often negative, equivalent to a loss, which is not desirable. All in all, it makes sense to conjecture that, one invests into the fund only if the case of death is ignored, or that bequest does not matter. This is taken as our assumption.

We start with applying the simple first-order differentiation technique and find that, fund members who are older respectively investing more into the fund earn a larger actuarial gain upon survival, and at the same time face a larger variance on the actuarial gain upon survival. In other words, these members are involuntarily facing a risk-return trade-off at a larger magnitude plainly due to the mechanism of the annuity overlay fund. Conversely, members who are younger respectively investing less face a smaller magnitude of risk-return trade-off.

Afterwards we attempt to address our question of whether the fund offers equal values to all fund members. Naturally, the answer depends very much on the members' personal valuation approaches. Different members can assign very

[1] Since all members can pick different investment strategies, if one is to consider also the randomness in market return, no analytical results can be established and one can only resort to numerical simulations.

different values to the same magnitude of trade-off simply because of different individual perceptions. Not only their relative weighting placed between risk and return matters, but their appreciation on wealth also plays a role. Among all, we are particularly interested in, how the members' attitude towards risks would influence their perception on the fund. To this end, we employ the expected utility theory, with which an individual's valuation on wealth is characterized by her own utility function, and her attitude towards risks is quantified by the absolute risk aversion (ARA) coefficient. With reference to these quantities, we take an approximation of the expected change in utility conditional on survival until terminal time as a metric of the fund's attractiveness, and conduct a comparison on the fund's attractiveness for two members with different wealth-mortality profiles.

The members' perception on their marginal increase in wealth as well as their attitude towards risks are crucial factors for the resulting conclusions. By assuming that the two members under comparison enjoy the same increase in utility per unit increase respectively per percentage increase in wealth, we give a numerical illustration on exploring the correlation between the risk aversion levels of the members and the relative attractiveness of the fund to the members.

Our purpose here is to give direction on examining the relative attractiveness of the fund between heterogeneous members at a high level. For the sake of generality, we do not assume any specific form of utility function and provide only a framework of the analysis. The analysis can be practically employed once the members' utility functions on wealth as well as their risk aversion levels are known.

1.4 Supporting Retirement Consumption

We also propose a method of operating the annuity overlay fund, with which the fund acts as an instrument for supporting retirement consumption.

The annuity overlay fund is advocated here, primarily because of its feature of being an open fund, an advantage that few other recently reviewed designs possess. This feature enables the fund to constantly admit new members into the fund, so that the total number of member remaining in the fund does not drop despite deceased members exiting, but remains stable as time passes. By the Law of Large Number, a stable instead of diminishing pool means members can receive gains at a relatively stable level of uncertainly at all time.

A brief description of our idea is given in the following.

We assume all individuals have no bequest motive and their mortality rates with respect to age remain unchanged. The fund operates on a periodical basis recursively, let us say one round of pooling every year, so that there is one payment made to participants at the end of each year. At the beginning of each year, certain fixed number of individuals attaining the retirement age enters the fund and must remain in the fund until either death or reaching some limiting age. Then, at each year-end, the wealth of the members who have perished is shared among all members who were in the fund at the beginning of the year, in accordance to the fund's sharing rule. For the surviving members, this random amount is directly paid out as the first part of their retirement benefit. Meanwhile, surviving members also withdraw some predetermined amount from her own fund account as the second part of the retirement benefit. In total, a surviving member obtains two sources of benefit at each year-end: first the random amount contributed from the perished members, second the known withdrawal amount from her own fund account.

In Chapter 4 where the proposed operation is discussed in details, our goal is to solve for the fixed withdrawal amount at each age for a participant, such that the retirement benefit for her, which is the sum of the two payments described above, is constant in expectation over time.

Under the proposed operation, by allowing randomness in the actual benefit payment amount, retirees bear the mortality risks themselves as a group, in exchange for the theoretically zero risk capital cost charged from their insurance premium. Comparing to an actuarially fair annuity, the expected retirement payment from the annuity overlay fund is always less, as there is also a small payment allocated to the deceased members. Yet, in the realistic case that risk capital costs are embedded in the annuity price (whereas there is none for the annuity overlay fund), it is possible that the annuity overlay fund is preferred over a traditional annuity, if the reduction in the expected payment from the fund is less than the loadings added to the annuity, and retirees are willing to bear some level of uncertainty in the actual benefit payment amounts.

In addition, we demonstrate that, in case of stochastic mortality, adaptations can be made to the proposed operation in the limiting case, so that the resulting stream of payment can remain constant in expectation.

1.5 Related Work

For readers who are interested, some literature relevant to the background discussed are summarized below for further references.

Cost-intransparency of Annuities

The cost-intransparency problem of annuities has been discussed by, for example, Friedman and Warshawsky (1988); Blake (1999); Mitchell et al. (1999) and Stewart (2007). Upon purchasing an annuity, the single premium paid by the annuitant is held and invested into the financial market by the insurer. The annuity price is calculated under some actuarial basis laid down by the insurer, which consists of assumptions on both investment returns and the population's mortality level. Mortality risk and investment risk are hence implicitly combined in the price determination process. From the perspective of an annuitant, all that are known are the quoted annuity price and the regular benefit payment to be received, whereas the many loadings involved behind, such as administration costs, investment fees, costs for bearing mortality risk and profit margin, are encapsulated as a black box. Such opaqueness of the loadings embedded in the annuity price has been suggested by, for example, Blake (1999) and Stewart (2007) to be one of the reasons for the observed low annuity demand.

Among academic literature, the cost of an annuity is often estimated by calculating the money's worth of the annuity, which is the ratio of the expected present value of all annuity payouts to the premium amount. The difference between the calculated money's worth and one is taken as an indicator for the percentage cost charged to the annuity. The values of the money's worth of annuities however differ significantly for different countries and different studies. For example, based on data in 1995, Mitchell et al. (1999) calculate the money's worth of annuity in the U.S. to be in the range of 74%-94%. Blake (1999) further decompose the charges into two categories, namely administrative costs and loadings due to heterogeneity of the population's survival probability, and has shown that each constitutes around half of the total costs. On the other hand, using data from 1998, Finkelstein and Poterba (2002) report a figure of money's worth in the U.K. of 99% for males, whereas Murthi et al. (2000) suggest 93%. Cannon and Tonks (2009) conducted a trend analysis and observed a dropping trend of money's worth from 90% to 85% from 1994 to 2007. In view of the significant variation in results in spite of the sophisticated mathematical methods employed in the relevant literature, it is not hard to imagine that for the general public, who have relatively less knowledge on financial topics, it is even harder to judge the value of an annuity product.

Other than the issue of being cost-intransparent, a variety of reasons have been suggested throughout the decades to explain the annuity puzzle. See e.g. Ramsay and Oguledo (2018) for an extensive summary on the huge amount of research on this topic.

Innovations pooling Mortality Risks

Along with the annuity overlay fund, various innovations which also work by pooling mortality risks of a group of individuals have been proposed in the literature. We list some of them below.

The tontine is a research topic currently gaining popularity. Its mechanism is simple: at inception a group of participants invests some amount of money, at each subsequent payment time, the interests of the investment from all participants are to be shared equally among the surviving participants. Originated centuries ago, this structure has been reviewed by Milevsky and Salisbury (2015), who reconstructed the term structure of the interest rate, under which participants can receive a much stabler stream of payment in expectation upon survival. A considerable amount of research on tontine has followed. For example, Milevsky and Salisbury (2016) extend the operation of tontine to heterogeneous participants, whereas Bernhardt and Donnelly (2019) propose a mechanism with which bequest can be provided to the deceased tontine participants. Bräutigam et al. (2017) compare the annuity overlay fund against the tontine. Extending from tontine, Chen et al. (2019) introduce the "tonnuity", which is a combination of tontine and annuity.

Piggott et al. (2005) propose and analyse the mathematical model of the group self-annuitization scheme, with which payouts to a participant mimic that from an annuity, but are adjusted according to the actual mortality experience of the group (lowered if mortality improves). Valdez et al. (2006) study the issues of demand and adverse selection of the scheme. Hanewald et al. (2013) show that the scheme plays a significant role in individual's choice of retirement strategy. The scheme is improved by Qiao and Sherris (2013) through allowing dynamic pooling and incorporating a stochastic mortality model.

The pooled annuity fund, proposed by Stamos (2008), works similarly as the tontine, with a major difference being that, wealth of the deceased is shared among the surviving members not equally but in proportion to their wealth in the fund. Under the expected utility framework, Stamos (2008) investigates the optimal investment and consumption strategies for homogeneous participants, a case which coincides with the tontine.

Donnelly (2015) analyses the annuity overlay fund together with the group self-annuitization scheme by Piggott et al. (2005) and the pooled annuity fund by Stamos (2008) from the perspective of actuarial fairness. It is shown that, with a pool of heterogeneous participants, only the annuity overlay fund is actuarially fair, meaning that each participant receives in expectation the same amount as what she has paid and no one in the pool subsidizes or benefits from the others.

1.6 Thesis Structure

The remainder of the thesis is structured as follows: Chapter 2 details the discrete-time model of the annuity overlay fund. In Chapter 3 we investigate the mean and variance features of the fund for members with different wealth-mortality profiles, then perform an analysis to examine the relative attractiveness of the fund between heterogeneous members. In Chapter 4 we propose a method to operate the annuity overlay fund for use in retirement aspect, with which participants obtain a constant stream of payment in expectation upon survival. Finally, Chapter 5 concludes the thesis. In view of the fairly large amount of notations to be defined in the thesis, a table of all notations to be used is provided in Appendix A for a clear overview.

Chapter 2

Discrete-Time Model

The mathematical model of the annuity overlay fund is detailed in this chapter. Extending from the continuous-time model in Donnelly et al. (2014), we develop the discrete-time model, which will be used throughout the remainder of the thesis.

2.1 Model Description

Let time 0 be the inception time and time $T > 0$ be the terminal time which is known. For example, T could be a month or a year. At time 0, there are $M \in \mathbb{N}$ surviving members in the annuity overlay fund, each member $m \in \{1, 2, \ldots, M\}$ invests some wealth amount $W_0^{(m)} > 0$ into the fund and has death probability ${}_TQ_0^{(m)}$ in the time interval $[0, T]$ without uncertainty. Denote the survival status of the member $m \in \{1, 2, \ldots, M\}$ at time T by $N_T^{(m)}$, where $N_T^{(m)} = 0$ means she is alive at time T and $N_T^{(m)} = 1$ otherwise.

At time 0, the wealth of each member is invested in the financial market, and each member decides independently on their own investment strategy. For an arbitrary member k, $R_T^{(k)}$ denotes the rate of market return on her wealth $W_0^{(k)}$ during the time interval $[0, T]$, which can be either certain or random depending on the member's investment strategy.[1] The new wealth $W_{T-}^{(k)}$ at time T after earning the market return is computed as $W_{T-}^{(k)} = W_0^{(k)} \left(1 + R_T^{(k)}\right)$.

The survival status and the market return of all fund members are defined on the same complete probability space $(\Omega, \mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_{T-}, \mathcal{F}_T\}, \mathbb{P})$. The filtrations $\mathcal{F}_0$ and $\mathcal{F}_T$ are respectively the σ-algebra generated by all random processes up to and including time 0 and T, which represent information on both the survival

[1] Here we have not established a market model, therefore the nature of $R_T^{(k)}$, whether being a constant or a random variable, is not specified. As market return is not being analysed, a detailed model for which is not required for establishing the subsequent contents, and we omit it to avoid adding unnecessary complexity to the thesis.

status and the market return of all members respectively at time 0 and T. The filtration $\mathcal{F}_{T-}$ is the same as $\mathcal{F}_T$ except that it excludes information on the members' survival status at time T.

For each member $m \in \{1, 2, \ldots, M\}$ in the fund, if she passes away during the time interval $[0, T]$, her wealth $W_{T-}^{(k)}$ is to be put into a notional mortality account. Hence, at time T, the total amount of money that has flown into the account due to all deaths in the interval $[0, T]$, denoted by U_T, is

$$U_T = \sum_{m=1}^{M} W_{T-}^{(m)} N_T^{(m)}. \tag{2.1}$$

This amount is then shared among all members who participated in the fund at time 0 (regardless of whether they survive or not at time T), in proportion to their wealth-mortality profile. An arbitrary member $k \in \{1, 2, \ldots, M\}$, who has probability of death ${}_TQ_0^{(k)}$ during the period $[0, T]$ and owns wealth $W_{T-}^{(k)}$ at time T, receives the fraction

$$\frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} \cdot U_T. \tag{2.2}$$

Denote by $G_T^{(k)}$ the actuarial gain obtained by member k at time T. If she survives until time T, she is awarded the amount given in (2.2). Otherwise, if she passes away during the interval $[0, T]$, she is still entitled to this amount given in (2.2), yet her wealth $W_{T-}^{(k)}$ has been forfeited and put into the notional mortality account at the first place, so she suffers from a loss of her initial wealth at the same time. Summarizing, this member earns an actuarial gain at time T according to her own survival status at time T, in accordance to the following:

$$G_T^{(k)} = \begin{cases} \dfrac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} \, U_T - W_{T-}^{(k)} & \text{, if the member dies during } [0, T], \\[2ex] \dfrac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} \, U_T & \text{, if the member is alive at time } T. \end{cases} \tag{2.3}$$

The only difference between the actuarial gain upon survival and death lies on the term $-W_{T-}^{(k)}$, which originates from the forfeited wealth of $W_{T-}^{(k)}$ upon the member's death during the period. Further, note that conditional on survival at the terminal time the actuarial gain is always non-negative, implying that upon survival a member never losses in terms of the actuarial gain.

At the end of the period, after distributing the actuarial gain, the pooling terminates. In total, at time T, the arbitrary member k owns the wealth accumulated with market return $W_{T-}^{(k)}$ plus the actuarial gain $G_T^{(k)}$ given in (2.3). That is, the new wealth of the member at time T after pooling, denoting by $W_T^{(k)}$, is

$$\begin{aligned} W_T^{(k)} =& W_{T-}^{(k)} + G_T^{(k)} \\ =& \begin{cases} \dfrac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} U_T & \text{, if the member dies during } [0,T], \\ W_{T-}^{(k)} + \dfrac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} U_T & \text{, if the member is alive at time } T. \end{cases} \end{aligned} \tag{2.4}$$

In short, the member always receives the fraction given in equation (2.2) as well as the market return independent of survival or death during the period, yet upon death her wealth $W_{T-}^{(k)}$ is to be given up.

Afterwards, another pooling can start again at time T, which is independent of the period $[0,T]$, by forming a new portfolio of members at time T. For example, previous members who survived the period $[0,T]$ can remain participating in the new period, and if so, they decide on some new amount to be invested into the fund for the new period. One can otherwise opt to exit. Clearly there can also be other new joiners into the fund. Based on the new portfolio of members at time T, a new round of pooling can be established in the same way as above.

The annuity overlay fund is distinct in that, it is pooling wealth of members only *periodically*. Consequently, individuals are free to enter and exit the fund as well as change their investment amount at the beginning of each period. This feature is made possible by that, the annuity overlay fund is actuarially fair for all members on a periodical basis, mathematically meaning

$$\mathbb{E}\left(G_T^{(k)} \middle| \mathcal{F}_{T-}\right) = 0 \tag{2.5}$$

for any k (for a detailed proof of equation (2.5) refer to Appendix B, which is a simple adaptation of the result in Donnelly et al. (2014) to discrete time frame).

In verbal terms, in each period, each member earns zero expected actuarial gain. At the end of a period, each member therefore does not owe or benefit from other members in expectation, and as a result is free to leave the fund. Further, individuals with any wealth-mortality profile are allowed to participate in the fund, as each individual's profile is being taken care of by fraction (2.2).

Remark 2.1 *Upon comparing with the model given in Donnelly et al. (2014), one would notice that here we express the portfolio of members in a slightly different manner. Instead of grouping members who have the same wealth-mortality profile into one unit as in the work by Donnelly et al. (2014), here we consider each member individually, or that each member forms one group by herself. The mechanism of the fund remains the same and all the mathematical formulas lead to the same results. We adopt this other form of expression, since in this way, each member can be treated independent of all other members. This is important for our analysis and will allow the subsequent contents to be followed more easily.*

2.2 Illustrative Example

The toy example given in Section 2 of Donnelly et al. (2014), which well illustrates the fund's operation on a discrete time basis, is cited below for clearer understanding. Additionally, to give a more complete picture, a market return is incorporated in the example below.

Suppose the time interval is as one month. At the beginning of the month, two individuals, Alice and Bob, participate in the annuity overlay fund. Each of the two has distinct wealth amount and probability of death during the month. The information on each of them is listed in Table 2.1.

Member	Wealth	Death probability in the month
Alice	1 000 000	0.2%
Bob	50 000	0.1%

Table 2.1: Information on Alice and Bob at the beginning of the month.

The wealth from each of Alice and Bob is to be invested in the financial market independently, according to their individually chosen investment strategy. Suppose at the end of the month, their investment strategy earns them an actual market return as given in Table 2.2.

Member	At the end of the month	
	Realized market return	Wealth before pooling
Alice	2%	$1\ 000\ 000 \times (1+2\%) = 1\ 020\ 000$
Bob	3%	$50\ 000 \times (1+3\%) = 51\ 500$

Table 2.2: Alice and Bob's realized market return at the end of the month.

If, for example, during the month Bob passes away, his wealth after rolling up with market return 51 500 is to be put into the notional mortality account. At the end of the month, Alice receives an actuarial gain from the death of Bob (according to equation (2.3)) of

$$51\ 500 \times \frac{1\ 020\ 000 \times 0.2\%}{1\ 020\ 000 \times 0.2\% + 51\ 500 \times 0.1\%} = 41\ 119,$$

whereas Bob receives an actuarial gain from his own death of

$$51\ 500 \times \frac{51\ 500 \times 0.1\%}{1\ 020\ 000 \times 0.2\% + 51\ 500 \times 0.1\%} - 51\ 500 = -41\ 119.$$

In total, the new wealth of Alice at the end of the month is (using equation (2.4)) $1\ 020\ 000 + 41\ 119 = 1\ 061\ 119$ and that of Bob is $51\ 500 - 41\ 119 = 10\ 381$. As Bob is dead, this amount is to be paid to his estate.

For all other possible scenarios, i.e. when only Alice perishes, both perish or both survives, calculations are done in the same way.

At the end of the month, after distributing the actuarial gain (if any), the pooling ends. Each of Alice and Bob, upon survival, can decide on whether to participate again in the annuity overlay fund in the next month, with the possibility of having also other new participants in the fund.

Chapter 3

Analysis on Fund Attractiveness

This chapter examines the relative attractiveness of the annuity overlay fund between members with different wealth-mortality profiles. The expected actuarial gain awarded to a member upon her survival at the terminal time, which we term as the expected "survival gain" for short in the sequel, is viewed as the return from the fund, whereas the variance of the survival gain is viewed as the risk. We apply first-order differentiation to examine the influence of a member's wealth and mortality information on her risk-return trade-off. To quantify the attractiveness of some risk-return trade-off offered by the fund, we approximate the expected change in utility resulting from the trade-off for a member. We then compare this quantity for members with different wealth-mortality profiles.

For simplicity in discussion, in the remainder we assume that an individual's death probability in a period is strictly increasing with age, that is, ${}_TQ_0^{(k_1)} < {}_TQ_0^{(k_2)}$ holds if and only if member k_1 is younger than member k_2.

3.1 Risk-Return Trade-off

Corresponding to the results in the original work by Donnelly et al. (2014), the expected survival gain for an arbitrary member k is

$$\mathbb{E}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)}=0\right) = {}_TQ_0^{(k)}W_{T-}^{(k)}\left(1-\frac{{}_TQ_0^{(k)}W_{T-}^{(k)}}{\sum_{m=1}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}\right), \tag{3.1}$$

and the variance of the survival gain is

$$\operatorname{Var}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)}=0\right) = \left(\frac{{}_TQ_0^{(k)}W_{T-}^{(k)}}{\sum_{m=1}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}\right)^2\left(\sum_{\substack{m=1,\\ m\neq k}}^{M}\left(W_{T-}^{(m)}\right)^2{}_TQ_0^{(m)}\right). \tag{3.2}$$

The detailed proofs for equations (3.1) and (3.2) are given respectively in Appendix C and D.

In the following, we understand how the expectation and variance of the rate of survival gain of a member, represented by equations (3.1) and (3.2), are influenced by the relevant factors, namely the fund's total number of members (a.k.a. the pool size) and all members' wealth-mortality profiles. In most cases, we adopt the simple analysing method of taking first order differentiation and observing the sign of change of the relevant quantity when a factor changes. Table 3.1 is the outline of the flow of observations to be made.

	Factors		
	Pool Size	Age	Wealth
Expectation/ Eq.(3.1)	Observation A	Observation B	Observation D
Variance/ Eq.(3.2)		Observation C	Observation E

Table 3.1: Flow of upcoming observations.

A. Effect of Pool Size

Let us take a closer look at equations (3.1) and (3.2). Obviously, the term $\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}$ increases with the pool size M, since $W_{T-}^{(m)}$ and ${}_TQ_0^{(m)}$ are strictly positive for all $m \in \{1, 2, \ldots, M\}$. Hence, the expected survival gain increases with pool size, whereas its variance decreases with pool size.

Assume that as the number of total fund members tends to infinity, the total wealth from all members also tends to infinity, which is a condition generally satisfied in reality. Analog to equations (11) and (12) in Donnelly et al. (2014), as the pool size tends to infinity, the expected survival gain tends to

$$\lim_{M\to\infty} \mathbb{E}\left(G_T^{(k)} \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0 \right) = {}_TQ_0^{(k)} W_{T-}^{(k)}, \tag{3.3}$$

or equivalently, the expected rate of survival gain tends to

$$\lim_{M\to\infty} \mathbb{E}\left(\frac{G_T^{(k)}}{W_{T-}^{(k)}} \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0 \right) = {}_TQ_0^{(k)}. \tag{3.4}$$

The limit on the variance is

$$\lim_{M\to\infty} \operatorname{Var}\left(G_T^{(k)} \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0\right) = 0. \tag{3.5}$$

As already discussed in Donnelly et al. (2014), with a finite pool size, the expected rate of survival gain of a member is always less than her own probability of death in the interval $[0, T]$. As the pool enlarges, all members benefit from an increased expected rate of survival gain with reduced uncertainty. With a pool size approaching infinity, every member earns survival gain at a rate that approaches her own death probability and uncertainty is eliminated. Participating in an annuity overlay fund with infinite pool is thus analogous to buying an insurance from insurance company, with which the insurer pays the insured a guaranteed rate of return that equals the insured's death probability conditional on survival. There is no randomness in the payment amount and no one else is involved.

B. Effect of Age on Expected Survival Gain

Consider an arbitrary member k of the fund. Taking differentiation on the expected survival gain w.r.t. her probability of death ${}_TQ_0^{(k)}$, we have

$$\frac{\partial}{\partial_T Q_0^{(k)}} \mathbb{E}\left(G_T^{(k)} \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0\right) = W_{T-}^{(k)} \left(1 - \frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}}\right)^2, \tag{3.6}$$

which is strictly positive, suggesting that an older member is allocated a higher expected rate of survival gain than a younger member, provided that both make the same investment. This is no surprise. Directly observing equation (3.1), a member's expected survival gain is always some fraction of the member's own death probability times her wealth, i.e. some fraction of the term ${}_TQ_t^{(k)} W_{T-}^{(k)}$. Older members, who have a higher death probability, straightforwardly obtain a higher expected survival gain.

C. Effect of Age on Variance of Survival Gain

Similarly, taking differentiation on the variance of the survival gain,

$$\begin{aligned}&\frac{\partial}{\partial_T Q_0^{(k)}}\mathrm{Var}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)}=0\right)\\ =&\frac{2\,{}_TQ_0^{(k)}\left(W_{T-}^{(k)}\right)^2\left(\sum_{m=1,m\neq k}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}\right)\left(\sum_{m=1,m\neq k}^{M}\left(W_{T-}^{(m)}\right)^2{}_TQ_0^{(m)}\right)}{\left(\sum_{m=1}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}\right)^3},\end{aligned} \tag{3.7}$$

which is strictly positive, indicating that an older member faces larger variance on the survival gain amount. Recall that conditional on survival at time T, the arbitrary member k earns the fraction $\frac{{}_TQ_0^{(k)}W_{T-}^{(k)}}{\sum_{m=1}^{M}{}_TQ_t^{(m)}W_{T-}^{(m)}}$ of the random amount that flows into the notional mortality account.The older the member is, the larger the fraction of the random amount in the notional mortality account is entitled as her survival gain, which straightforwardly means the member is bearing more uncertainty in the survival gain amount.

Combining observations B and C, older members in the fund are allocated a higher return and at the same time face a higher risk than younger members. In other words, older members are involuntarily facing a risk-return trade-off at a larger magnitude.

D. Effect of Wealth on Expected Survival Gain

Taking differentiation on the expected survival gain w.r.t. the wealth of member k yields

$$\begin{aligned}&\frac{\partial}{\partial W_{T-}^{(k)}}\mathbb{E}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)}=0\right)\\ =&\frac{{}_TQ_0^{(k)}W_{T-}^{(k)}\sum_{m=1,m\neq k}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}{\sum_{m=1}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}\left(1-\frac{{}_TQ_0^{(k)}}{\sum_{m=1}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}\right),\end{aligned} \tag{3.8}$$

which is strictly positive. That is, a member who invests more wealth into the fund is allocated a higher expected survival gain than a member who invests less, given that both are of the same age. This is intuitively true, an individual who invests more earns naturally a higher return in absolute terms.

On the other hand, if we consider the rate of survival gain instead of the absolute amount of survival gain, we obtain

$$\frac{\partial}{\partial W_{T-}^{(k)}}\mathbb{E}\left(\frac{G_T^{(k)}}{W_{T-}^{(k)}}\middle|\mathcal{F}_{T-}, N_T^{(k)}=0\right) = \frac{-\left({}_TQ_0^{(k)}\right)^2\sum_{m=1,m\neq k}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}{\left(\sum_{m=1}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}\right)^2}, \quad (3.9)$$

which is strictly negative, implying that a member who invests more is allocated a smaller rate of survival gain than a member who invests less. The contrasting results from equations (3.8) and (3.9) is explained by that, as the wealth of a member increases, while her survival gain increases accordingly, the rate of increase of her survival gain is slower than the rate of increase of her wealth itself.

E. Effect of Wealth on Variance of Survival Gain

Lastly, take differentiation on the variance w.r.t. the wealth of member k,

$$\frac{\partial}{\partial W_{T-}^{(k)}}\mathrm{Var}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)}=0\right)$$

$$= \left(\frac{{}_TQ_0^{(k)}W_{T-}^{(k)}}{\sum_{m=1}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}\right)^2\left(\sum_{\substack{m=1,\\ m\neq k}}^{M}\left(W_{T-}^{(m)}\right)^2{}_TQ_0^{(m)}\right)\left(1-\frac{2\cdot{}_TQ_0^{(m)}W_{T-}^{(k)}}{\sum_{m=1}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}\right). \quad (3.10)$$

The expression in equation (3.10) is positive if the wealth of member k satisfies $W_{T-}^{(k)} < \frac{\sum_{m=1,m\neq k}^{M}{}_TQ_0^{(m)}W_{T-}^{(m)}}{{}_TQ_0^{(k)}}$. Note that this is a condition that generally holds in practice, as long as the pool is not too small and member k is not very much older than all other members. (For instance, such condition would be very unlikely to be violated with a pool of a hundred members.) Under this mild condition, a member who invests more wealth faces a larger variance on her survival gain amount. In the remainder of this chapter we consider only this usual case.

From observations D and E we come up with two cases. The first case is to consider return as the expected amount of the survival gain. As members who invest more wealth into the fund are awarded a higher expected survival gain along with a larger variance on the gain, members can therefore influence the magnitude of the risk-return trade-off through decision on the wealth amount to be invested. By investing more, members face a risk-return trade-off at a larger magnitude, vice versa. The second case is that one instead considers return in terms of the expected rate of survival gain. As shown above, members who invest more wealth are entitled to a lower return, despite their risk being higher. In this regard, investing less is always more favourable.

The observations above are summarized in Table 3.2.

	Factors		
	Increase in Pool Size	Increase in Age	Increase in Wealth
Expected survival gain	Increase	Increase	Increase
Expected rate of survival gain	Increase	Increase	Decrease
Variance of survival gain	Decrease	Increase	Increase in usual cases

Table 3.2: Summary of observations.

3.2 Expected Change in Utility

From the observations above, members of the annuity overlay fund with different wealth-mortality profiles face some risk-return trade-off at different magnitudes. Our question is, given individuals' different profiles, and hence the corresponding different magnitudes of the trade-off, whether certain groups would find the fund more attractive than the others. For example, given the same wealth amount is invested, an older individual might value the fund less than a younger individual, if she is more concerned with the higher risk that she faces; or that the older individual might value the fund more than the younger individual, if she puts more value on the higher expected return that she receives. This depends, among others, on the members' perception on wealth as well as their attitude towards risks.

To quantify our problem, we conduct an analysis under the framework of expected utility, where an individual's valuation on wealth is described by her utility function on wealth, and her risk-aversion level can be represented by the absolute risk aversion (ARA) coefficient.

Denote the utility function of an arbitrary member k to be $u_k(\cdot)$, so that $u_k(W)$ is the utility offered to the individual from some wealth amount W. We consider the expected change in utility for an arbitrary member k due to pooling risks through the fund. For member k, who invests in the annuity overlay fund in the time interval $[0,T]$, her wealth at time T before pooling risks is W_{T-} and that after pooling risks is W_T. Given information at time 0 and survival at time T, the concerned quantity is

$$\mathbb{E}\left[u_k\left(W_T^{(k)}\right) - u_k\left(W_{T-}^{(k)}\right)\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right]. \tag{3.11}$$

Recall that the survival gain for a fund member is always non-negative, i.e. $\left\{W_T^{(k)} - W_{T-}^{(k)}\middle|N_T^{(k)} = 0\right\} \geq 0$, hence her expected change in utility condition on survival is also always non-negative. With this in mind, from here on we use the terms expected change in utility and expected increase in utility interchangeably. If the expected increase in utility for one member is greater than that for another member, then the former member would find the fund more attractive, in the sense that she benefits from a larger increase in utility in expectation, vice versa.

Denote by $u_k'(W)$ and $u_k''(W)$ respectively the first and second order derivative of $u_k(W)$ w.r.t. W.

Proposition 3.1 *Given information at time 0 and survival upon the terminal time T, the expected change in utility for member k due to pooling risks in the annuity overlay fund can be approximated using the mean and variance of the survival gain amount, by*

$$\begin{aligned}&\mathbb{E}\left[u_k\left(W_T^{(k)}\right) - u_k\left(W_{T-}^{(k)}\right)\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right]\\ \approx &u_k'\left(W_{T-}^{(k)}\right)\left[\mathbb{E}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right) - \frac{A_k\left(W_{T-}^{(k)}\right)}{2}\mathrm{Var}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right)\right]\end{aligned} \tag{3.12}$$

where $A_k(W) := -\frac{u_k''(W)}{u_k'(W)}$ is the ARA coefficient when member k has wealth W, and $\mathbb{E}\left(G_T^{(k)}|\mathcal{F}_{T-}, N_T^{(k)} = 0\right)$ and $\mathrm{Var}\left(G_T^{(k)}|\mathcal{F}_{T-}, N_T^{(k)} = 0\right)$ are as given in equations (3.1) and (3.2).

Proof. Applying Taylor expansion up to second order on $u_k\left(W_T^{(k)}\right)$ around $W_{T-}^{(k)}$,

$$
\begin{aligned}
&u_k\left(W_T^{(k)}\right)\\
\approx &u_k\left(W_{T-}^{(k)}\right)+u_k'\left(W_{T-}^{(k)}\right)\left(W_T^{(k)}-W_{T-}^{(k)}\right)+\frac{1}{2}u_k''\left(W_{T-}^{(k)}\right)\left(W_T^{(k)}-W_{T-}^{(k)}\right)^2.
\end{aligned}
$$

Note that the goodness of the approximation depends on the magnitude of $W_T^{(k)} - W_{T-}^{(k)}$. The omitted higher order terms are small, as long as there is not a large amount of death in the pool in the time interval $[0,T]$ such that the difference between $W_T^{(k)}$ and $W_{T-}^{(k)}$ is small. Taking expectation conditional on information at time 0 and on survival at time T,

$$
\begin{aligned}
&\mathbb{E}\left[u_k\left(W_T^{(k)}\right)-u_k\left(W_{T-}^{(k)}\right)\middle|\mathcal{F}_{T-},N_T^{(k)}=0\right]\\
\approx &u_k'\left(W_{T-}^{(k)}\right)\mathbb{E}\left(W_T^{(k)}-W_{T-}^{(k)}\middle|\mathcal{F}_{T-},N_T^{(k)}=0\right)\\
&+\frac{1}{2}u_k''\left(W_{T-}^{(k)}\right)\mathbb{E}\left[\left(W_T^{(k)}-W_{T-}^{(k)}\right)^2\middle|\mathcal{F}_{T-},N_T^{(k)}=0\right]\\
=&u_k'\left(W_{T-}^{(k)}\right)\mathbb{E}\left(G_T^{(k)}\middle|\mathcal{F}_{T-},N_T^{(k)}=0\right)+\frac{1}{2}u_k''\left(W_{T-}^{(k)}\right)\operatorname{Var}\left(G_T^{(k)}\middle|\mathcal{F}_{T-},N_T^{(k)}=0\right)\\
=&u_k'\left(W_{T-}^{(k)}\right)\left[\mathbb{E}\left(G_T^{(k)}\middle|\mathcal{F}_{T-},N_T^{(k)}=0\right)+\frac{u_k''\left(W_{T-}^{(k)}\right)}{2u_k'\left(W_{T-}^{(k)}\right)}\operatorname{Var}\left(G_T^{(k)}\middle|\mathcal{F}_{T-},N_T^{(k)}=0\right)\right].
\end{aligned}
$$

□

Note that $u_k'\left(W_{T-}^{(k)}\right)$, which represents the marginal utility for member k from her wealth $W_{T-}^{(k)}$, is positive for any $W_{T-}^{(k)} > 0$ with a sensible utility function. Further, assuming member k is risk-averse, then her ARA coefficient $A_k\left(W_{T-}^{(k)}\right)$ is also positive for any wealth $W_{T-}^{(k)} > 0$. Approximation (3.12) can thus be considered separately in two parts as follows: the first term is the expected *increase* in utility rewarded by the expected survival gain, whereas the second term is the expected *decrease* in utility arisen from the variance of the survival gain.

The magnitude of the overall expected change in utility depends on the magnitude of the ARA coefficient $A_k\left(W_{T-}^{(k)}\right)$. When $A_k\left(W_{T-}^{(k)}\right)$ is small, meaning member k is less risk averse and variation on the survival gain amount matters less to her, the expected increase in utility is higher, indicating a higher attractiveness of the fund to member k. Contrarily, when $A_k\left(W_{T-}^{(k)}\right)$ is large, meaning member k is more sensitive to the uncertainty on the survival gain, her expected increase in utility is smaller and the fund appears less attractive to her. Using the ARA coefficient as a measure of an individual's attitude towards risk, we compare the expected increase in utility for fund members with different risk-mortality profiles under different risk appetites.

To proceed with our analysis, we consider two members in the fund, k_1 and k_2, who have respectively probability of death ${}_TQ_0^{(k_1)}$ and ${}_TQ_0^{(k_2)}$ in the time interval $[0,T]$, and own $W_{T-}^{(k_1)}$ and $W_{T-}^{(k_2)}$ at time T before pooling risks through the fund. Further, member k_1 values wealth according to the utility function $u_{k_1}(W)$, and whose attitude towards risk given some wealth W is characterized by the ARA coefficient $A_{k_1}(W)$, whereas member k_2 values wealth according to the utility function $u_{k_2}(W)$ and whose ARA coefficient given some wealth W is $A_{k_2}(W)$.

Proposition 3.2 *Member k_1 and member k_2 obtain equal expected change in utility through participating in the fund in the time interval $[0,T]$ if and only if*

$$A_{k_1}\left(W_{T-}^{(k_1)}\right) = f \cdot A_{k_2}\left(W_{T-}^{(k_1)}\right) + g, \tag{3.13}$$

where

$$f = \frac{u'\left(W_{T-}^{(k_2)}\right)}{u'\left(W_{T-}^{(k_1)}\right)} \cdot \frac{\operatorname{Var}\left(G_T^{(k_2)}\middle|\mathcal{F}_{T-}, N_T^{(k_2)}=0\right)}{\operatorname{Var}\left(G_T^{(k_1)}\middle|\mathcal{F}_{T-}, N_T^{(k_1)}=0\right)} \text{ and}$$

$$g = \frac{2\cdot\mathbb{E}\left(G_T^{(k_1)}\middle|\mathcal{F}_{T-}, N_T^{(k_1)}=0\right)}{\operatorname{Var}\left(G_T^{(k_1)}\middle|\mathcal{F}_{T-}, N_T^{(k_1)}=0\right)} - \frac{u'\left(W_{T-}^{(k_2)}\right)}{u'\left(W_{T-}^{(k_1)}\right)} \cdot \frac{2\cdot\mathbb{E}\left(G_T^{(k_2)}\middle|\mathcal{F}_{T-}, N_T^{(k_2)}=0\right)}{\operatorname{Var}\left(G_T^{(k_1)}\middle|\mathcal{F}_{T-}, N_T^{(k_1)}=0\right)}.$$

The curve (3.13), which is a straight line on the $A_{k_1}\left(W_{T-}^{(k_1)}\right) - A_{k_2}\left(W_{T-}^{(k_2)}\right)$ plane, is termed as the "equivalence curve" for members k_1 and k_2 in the sequel. If $A_{k_1}\left(W_{T-}^{(k_1)}\right) < f\cdot A_{k_2}\left(W_{T-}^{(k_2)}\right)+g$, member k_1 obtains a higher expected change in utility then member k_2 and hence would find the fund more attractive than member k_2. If $A_{k_1}\left(W_{T-}^{(k_1)}\right) > f\cdot A_{k_2}\left(W_{T-}^{(k_2)}\right)+g$, the opposite happens.

Proof. For members k_1 and k_2 to have the same expected change in utility, use

Proposition 3.1 and equate the expressions for the two members, that is,

$$
\begin{aligned}
&u'_{k_1}\left(W_{T-}^{(k_1)}\right) \\
&\cdot \left[\mathbb{E}\left(G_T^{(k_1)}\middle|\mathcal{F}_{T-}, N_T^{(k_1)}=0\right) - \frac{A_{k_1}\left(W_{T-}^{(k_1)}\right)}{2}\mathrm{Var}\left(G_T^{(k_1)}\middle|\mathcal{F}_{T-}, N_T^{(k_1)}=0\right)\right] \\
=&u'_{k_2}\left(W_{T-}^{(k_2)}\right) \\
&\cdot \left[\mathbb{E}\left(G_T^{(k_2)}\middle|\mathcal{F}_{T-}, N_T^{(k_2)}=0\right) - \frac{A_{k_2}\left(W_{T-}^{(k_2)}\right)}{2}\mathrm{Var}\left(G_T^{(k_2)}\middle|\mathcal{F}_{T-}, N_T^{(k_2)}=0\right)\right].
\end{aligned}
$$

Rearranging terms gives the statement.

Similarly, for member k_1 to obtain a higher(lower) expected change in utility than member k_2, replace the equality sign by the greater-than(smaller-than) sign, then rearrange terms. (The assumption that the marginal utility $u'(W)$ is strictly positive for any W applies.) □

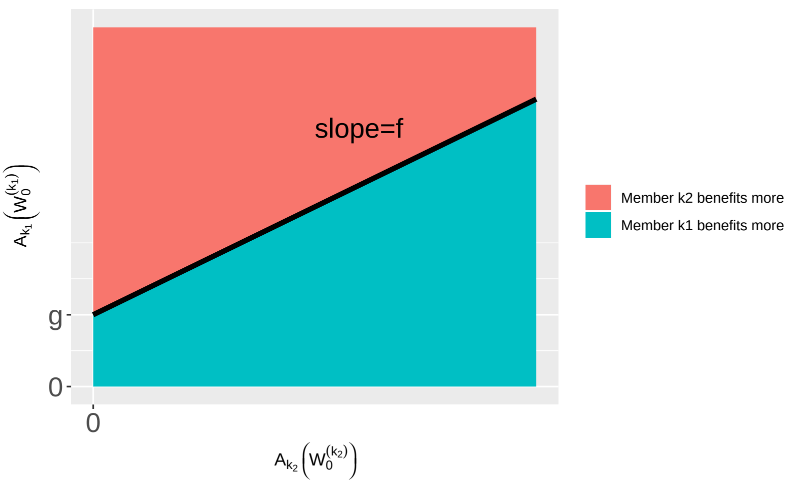


Figure 3.1: Example equivalence curve (black line).

An example of the equivalence curve on the $A_{k_1}\left(W_{T-}^{(k_1)}\right) - A_{k_2}\left(W_{T-}^{(k_2)}\right)$ plane proposed in Proposition 3.2 is sketched in Figure 3.1. An equivalence curve has a slope of f, which is strictly positive, and intercepts the y-axis at g. When the risk aversion levels of the two members k_1 and k_2, represented by $A_{k_1}\left(W_{T-}^{(k_1)}\right)$ and $A_{k_2}\left(W_{T-}^{(k_2)}\right)$, fall on the equivalence curve, both members benefits equally from the fund, in the sense that both obtain the same expected change in utility.

If the combination of the ARA coefficients of the two members lies above the equivalence curve, member k_1 benefits more than member k_2 in terms of the expected change in utility, vice versa.

As we see, the equivalence curve given in Proposition 3.2 depends on the marginal utility from wealth of one member relative to that of the other member, represented by the ratio of the terms $u'_{k_1}\left(W_{T-}^{(k_1)}\right)$ and $u'_{k_2}\left(W_{T-}^{(k_2)}\right)$ in the expressions of f and g.

Two specific cases will be considered in the numerical illustrations in Section 3.3, for which we give a description below.

Case 1: Same increase in utility per unit increase in wealth

The first case is that, the two members k_1 and k_2 benefits from an equal increase in utility per *unit* increase in their respective wealth, or that their marginal utility from their respective wealth amount is the same. This is represented by $u'_{k_1}\left(W_{T-}^{(k_1)}\right) = u'_{k_2}\left(W_{T-}^{(k_2)}\right)$.

Recall that the first term in approximation (3.12) is the expected increase in utility rewarded by the expected survival gain. This term has the coefficient being $u'_k\left(W_{T-}^{(k)}\right)$, the marginal utility from wealth. For different members who value wealth in different ways, namely assigning different marginal utility to their wealth, the same expected survival gain from the fund would reward them different expected increase in utility.

In the specific case that the two members k_1 and k_2 assign the same marginal utility to their wealth, their expected increase in utility contributed from the expected survival gain can be compared by directly comparing the magnitude of their expected survival gain. Further, if the expected survival gain for the two members are also the same, then the difference between their overall expected change in utility is determined by the difference in the second term of approximation (3.12), namely the risk aversion level as well as the variance on the survival gain of the two members.

Case 2: Same increase in utility per percentage increase in wealth

The second case is that, the two members k_1 and k_2 benefit from an equal increase in utility per *percentage* increase in their respective wealth. Mathematically, this is $u'_{k_1}\left(W_{T-}^{(k_1)}\right) \cdot W_{T-}^{(k_1)} = u'_{k_2}\left(W_{T-}^{(k_2)}\right) \cdot W_{T-}^{(k_2)}$. For instance, suppose member k_1 has wealth 1000 and member k_2 has wealth 2000, the two members benefit from the same increase in utility if member k_1's wealth is increased by 10 and that of member k_2 is increased by 20.

Again, take look at the first term of approximation (3.12). This term can be viewed as

$$\begin{aligned} & u'_k\left(W_{T-}^{(k)}\right)\mathbb{E}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)}=0\right) \\ =& u'_k\left(W_{T-}^{(k)}\right)\cdot W_{T-}^{(k)}\cdot\mathbb{E}\left(\frac{G_T^{(k)}}{W_{T-}^{(k)}}\middle|\mathcal{F}_{T-}, N_T^{(k)}=0\right). \end{aligned} \tag{3.14}$$

This is to say, in case two members k_1 and k_2 benefit from the same increase in utility per percentage increase in wealth, then their expected increase in utility contributed from the expected survival gain can be compared by simply considering their expected *rate of survival gain*. Further, if the expected rate of survival gain for the two members are equal, then the difference between their overall expected change in utility is determined again only by the risk aversion level and variance on the survival gain of the two members.

3.3 Numerical Illustrations

3.3.1 Mortality Model

Germany is taken as the reference country and its data is used for the subsequent parametrizations and illustrations.

For the mortality model, we employ the Gompertz model, under which the force of mortality of an individual at age x, denoted by λ_x, has the form

$$\lambda_x = \frac{1}{b}\exp\left(\frac{x-m}{b}\right), \tag{3.15}$$

where $b, m > 0$ are some constants. Here the parameter m is the modal age at death, whereas b is regarded as the dispersion coefficient.

The model is fitted to the 2017 data on German population at ages 65-110 available from the Human Mortality Database (HMD). For parametrization, death rate is assumed to follow Poisson distribution, and correspondingly the loss function is set as

$$-\sum \left(\text{Death count} \cdot \log \lambda_x - \text{Exposure} \cdot \lambda_x\right). \tag{3.16}$$

Upon minimizing the loss function, the model is parametrized with $m = 88.13$ and $b = 8.66$. The fitted mortality curve together with relevant information on the fitted model are given in Figure 3.2.

3.3.2 Equivalence Curves

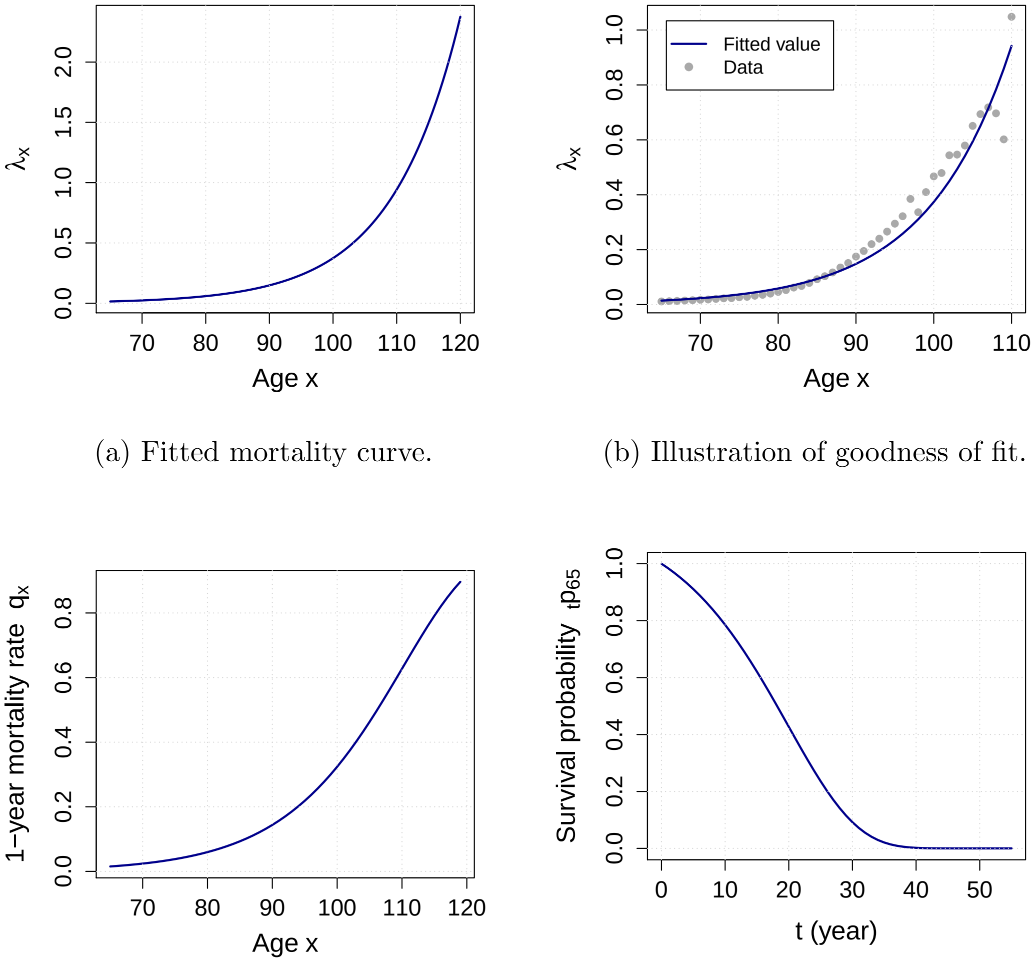
The portfolio of members is set as following: there exist 10 members at each of the age in the range 65-94, the total number of members is hence $M = 300$. All members owns a wealth amount of 1 000 at time T before pooling risks unless otherwise specified.

By inspecting the equivalence curve, we examine the interaction between the ARA coefficients of two members, k_1 and k_2, with which either one would benefit from the fund equally or differently than the other member in terms of the expected change in utility.

Age Effect

The first part of the example considers that the two members k_1 and k_2 are at different ages and invest the same amount of wealth 1 000 into the fund. We fix member k_2 to be at age 65. Further, we consider the case discussed on pages 27-28: both members assign the same marginal utility to their wealth.[1] In Figure 3.3a are the resulting equivalence curves on the $A_{k_1}\left(W_{T-}^{(k_1)}\right) - A_{k_2}\left(W_{T-}^{(k_2)}\right)$ plane upon varying the age of member k_1 from 65 to 94. Figures 3.3b and 3.3c separate

[1] Since here both members invest the same amount of wealth, the discussed case 1 and case 2 on pages 27-28 are equivalent.



(a) Fitted mortality curve.

(b) Illustration of goodness of fit.

(c) One-year mortality rate w.r.t. age.

(d) Survival probability w.r.t. time for an individual aged 65.

Figure 3.2: Information on fitted mortality model.

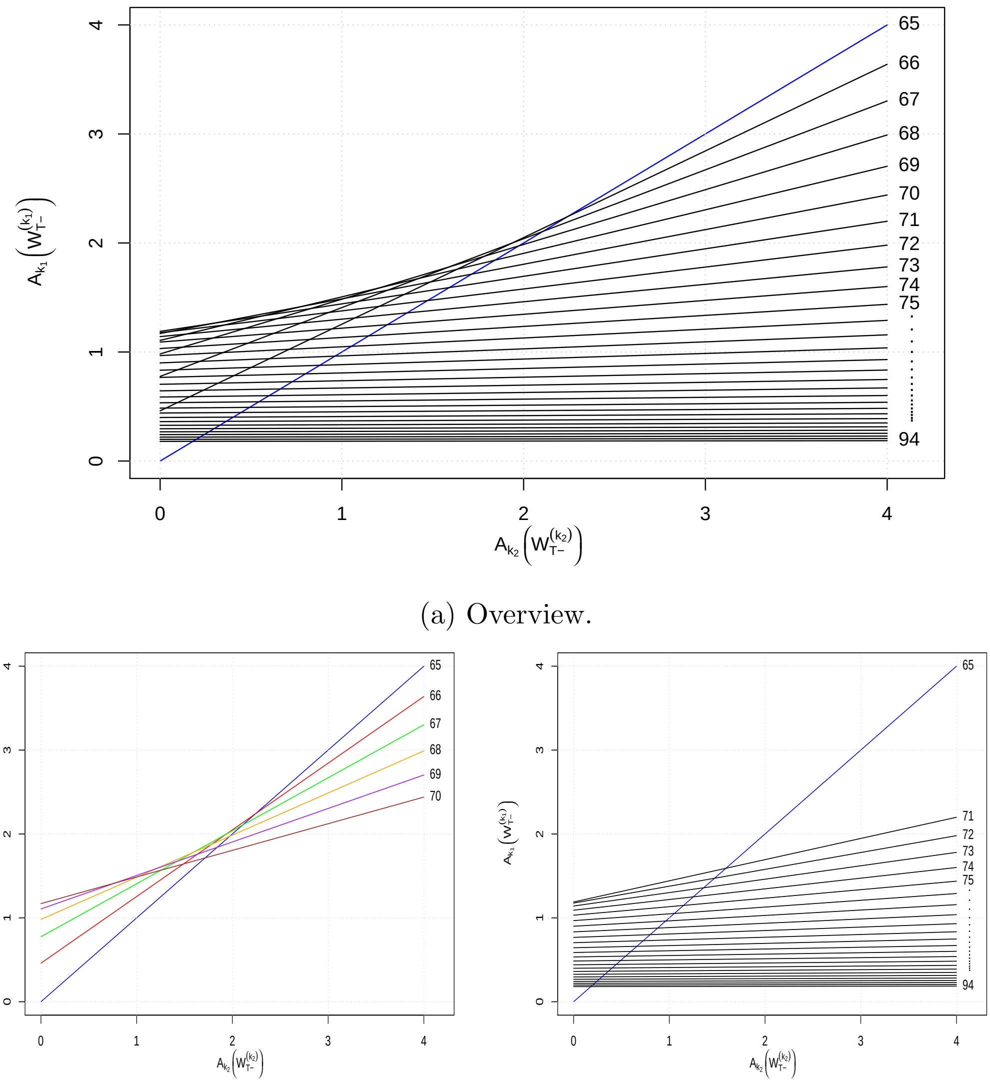
the equivalence curves in Figure 3.3a into two groups, which are provided for a clearer inspection.

To understand Figure 3.3a, we start from the blue line, which is the equivalence curve when both members k_1 and k_2 are at age 65 and we take this as the reference line. Since the two members have the same wealth-mortality profile, their expected survival gain from the fund as well as the variance on the survival gain to be faced are identical. In this case, the two members experience the same expected change in utility if and only if the two members have the same attitude towards risk, characterized by the same ARA coefficient. Therefore the equivalence curve is simply the straight line $A_{k_1}\left(W_{T-}^{(k_1)}\right) = A_{k_2}\left(W_{T-}^{(k_2)}\right)$. If, for example, member k_1 is less risk averse than member k_2, that is, $A_{k_1}\left(W_{T-}^{(k_1)}\right) < A_{k_2}\left(W_{T-}^{(k_2)}\right)$, corresponding to the region underneath the blue equivalence curve, member k_1 is less sensitive to the variance of the survival gain and hence she benefits from a higher expected increase in utility through participating in the fund than member k_2.

Then, consider now that the two members are of different ages. Suppose member k_1 is one year older, being at age 66. The corresponding equivalence curve is coloured red in Figure 3.3b for easy inspection. As shown, the red equivalence curve is less steep and intersects with the blue referencing curve at a point close to $(2.24, 2.24)$.

In our current setting where $u'_{k_1}\left(W_{T-}^{(k_1)}\right) = u'_{k_2}\left(W_{T-}^{(k_2)}\right)$, the slope of the equivalence curve is simply the ratio of the variance of survival gain of the two members under consideration (refer to equation (3.13)). Comparing to the blue reference line, the flatter slope of the red equivalence curve is a manifestation of the fact that, relative to member k_2, now the variance of survival gain faces by member k_1 becomes larger.

The flatter red equivalence curve from Figure 3.3b can be comprehended in two parts. Suppose both members k_1 and k_2 are slightly risk averse, say the ARA coefficients for both are below 2.24. Both members are less concerned with the variation of the survival gain amount, and the overall expected change in utility is driven mainly by the non-negative expected survival gain. It is known from Section 3.1 observation B that, member k_1, who is older than member k_2, is rewarded a higher expected survival gain from the fund. Hence, given the same risk aversion level for both members, member k_1 benefits more in terms of utility



(a) Overview.

(b) Selected equivalence curves with member k_1's age from 65 to 70.

(c) Selected equivalence curves with member k_1's age from 71 to 94.

Figure 3.3: Equivalence curves for members k_1 and k_2, assuming both investing a wealth of 1000 and member k_2 is 65 years old. The numbers on the right are the assumed age of member k_1 for the corresponding equivalence curves.

than member k_2. In other words, for both members to obtain the same expected change in utility, member k_1 must be more risk averse than member k_2, so that she is more sensitive to the variation of the survival gain, reflected by that, the red line is above the blue line in the region $A_{k_1}\left(W_{T-}^{(k_1)}\right), A_{k_2}\left(W_{T-}^{(k_2)}\right) < 2.24$. On the contrary, if both members are rather risk averse, where their ARA coefficients are greater than 2.24, both members are now more concerned with the uncertainty of the survival gain amount. From Section 3.1 Observation C, the older member k_1 faces a larger variance on the survival gain amount. Therefore, for both members to experience equal expected change in utility, member k_1 must be less risk averse than member k_2, so that she is less sensitive to the higher variation of the survival gain, reflected by that, the red line is below the blue line in the region $A_{k_1}\left(W_{T-}^{(k_1)}\right), A_{k_2}\left(W_{T-}^{(k_2)}\right) > 2.24$.

Similar arguments hold when member k_1 is further older, being at age 67, 68, 69 and 70. See Figure 3.3b for the various equivalence curves highlighted in different colours. The older the member k_1 is, the flatter the equivalence curve becomes. For example, when member k_1 is 67-years-old, the corresponding equivalence curve, which is the green curve, shifts further up for $A_{k_1}\left(W_{T-}^{(k_1)}\right), A_{k_2}\left(W_{T-}^{(k_2)}\right)$ small, which is due to the higher expected survival gain; and it shifts further down for $A_{k_1}\left(W_{T-}^{(k_1)}\right), A_{k_2}\left(W_{T-}^{(k_2)}\right)$ large, which is due to the higher variance on the survival gain.

As member k_1 turns even older, after age 71, the equivalence curve shifts downward for all $A_{k_2}\left(W_{T-}^{(k_2)}\right)$. See Figure 3.3c. This is explained by that, crossing the age 70, the uncertainty on the survival gain faced by member k_1 is growing quickly enough that, comparing to being one year younger, member k_1 must always be less risk averse so that she and member k_2 experience the same expected change in utility.

Wealth Effect

We proceed to the next part, where the two members k_1 and k_2 are of the same age and invest different amounts of wealth into the fund. Members k_1 and k_2 are both chosen to be at age 80, and member k_2 invests 1 000 into the fund. We consider the two special cases presented on pages 27-28 separately.

Case 1 Suppose the increase in utility per *unit* increase in initial wealth is the same for both members k_1 and k_2 that is, $u'_{k_1}\left(W_{T-}^{(k_1)}\right) = u'_{k_2}\left(W_{T-}^{(k_2)}\right)$. Figure 3.4 is the equivalence curve upon varying the wealth from member k_1.

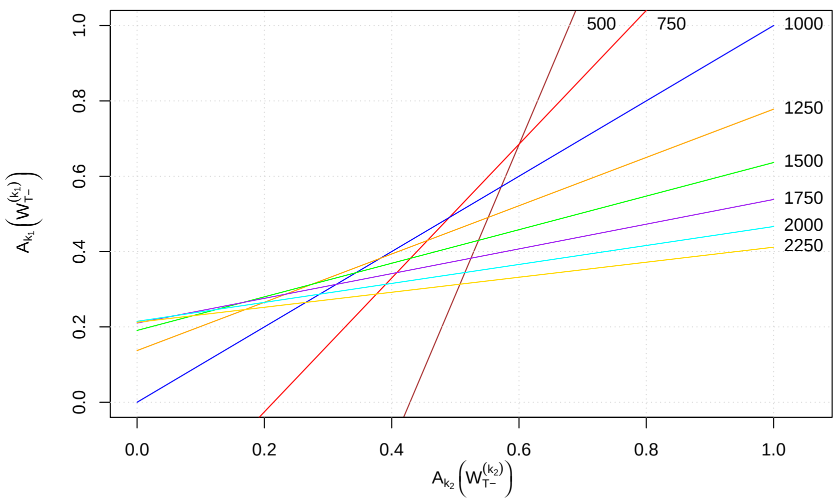


Figure 3.4: Equivalence curves for members k_1 and k_2, assuming both are 80 years old, member k_2 invests 1 000 and the two members enjoy the same increase in utility per unit increase in wealth. The numbers on the right are the assumed wealth of member k_1 for the corresponding equivalence curves.

To begin with, we have again the blue line, $A_{k_1}\left(W_{T-}^{(k_1)}\right) = A_{k_2}\left(W_{T-}^{(k_2)}\right)$, as the reference, which is the equivalence curve when the wealth of member k_1 is 1 000, i.e. the two members' wealth-mortality profiles are identical.

As illustration, take the case that member k_1 invests 750 whereas member k_2 invests 1 000. The resulting equivalence curve is the red line in Figure 3.4. Recall from Section 3.1 observations D and E that, a member who invests less into the fund is allocated a smaller expected survival gain from the fund and at the same time faces a smaller variance on the survival gain amount. Comparing with the blue reference line, it can be seen that, if both members are slightly risk averse, with $A_{k_1}\left(W_{T-}^{(k_1)}\right), A_{k_2}\left(W_{T-}^{(k_2)}\right) < 0.49$, the red line lies below the blue line. The same argument as that in the discussion for Figure 3.3a applies. Among the expectation and variance of survival gain, both members are relatively less sensitive to the variance and more concerned with the expected survival gain, while member k_1 earns a lower expected survival gain. Therefore, she has to be less risk averse than member k_2 in order that both experience the same expected change in utility. If both members are more sensitive to risk, with $A_{k_1}\left(W_{T-}^{(k_1)}\right), A_{k_2}\left(W_{T-}^{(k_2)}\right) > 0.49$, since member k_1 faces a smaller variance, she must be more risk averse than member k_2 such that both experience the same expected change in utility.

In the contrary case that member k_1 invests more than member k_2 into the fund, the reverse arguments hold and this is represented by the flatter slope of the corresponding resulting equivalence curves.

Case 2 Now suppose the increase in utility per *percentage* increase in initial wealth is the same for both members k_1 and k_2, that is, $u'_{k_1}(W_{T-}^{(k_1)}) \cdot W_{T-}^{(k_1)} = u'_{k_2}(W_{T-}^{(k_2)}) \cdot W_{T-}^{(k_2)}$. Figure 3.5 are the equivalence curves upon varying the wealth from member k_1 in this case.

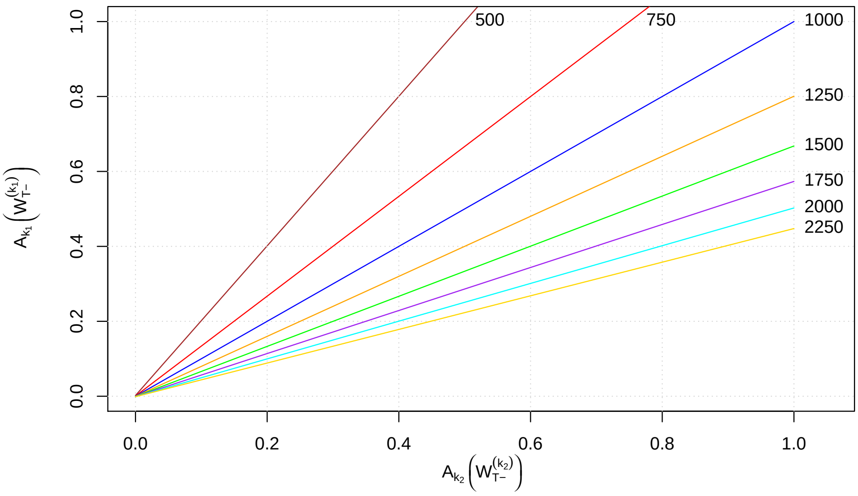


Figure 3.5: Equivalence curves for members k_1 and k_2, assuming both are 80 years old, member k_2 invests 1 000 and the two members enjoy the same increase in utility per percentage increase in wealth. The numbers on the right are the assumed wealth of member k_1 for the corresponding equivalence curves.

Here we obtain a graph very different from that in case 1. All equivalence curves starts from the origin, which is the situation that both members k_1 and k_2 are risk-neutral. If the two members are risk averse, there is no crossing of curves, no matter how much member k_1 invests.

In the current scenario, as already discussed on page 28, the difference in the part of expected increase in utility contributed from the return is caused only by a different expected rate of survival gain. Let us again take the case that member k_1 invests 750 and member k_2 invests 1 000 as illustration. Again from observation D in Section 3.1, now that member k_1 invests less than member k_2 into the fund, she therefore earns a higher expected rate of survival gain than member k_2. At the same time, member k_1 faces a smaller variance of the survival gain amount. A higher expectation together with a smaller variance straightforwardly implies that, the fund is favouring member k_1 in both aspects. Therefore, the two members can experience the same expected change in utility only if the risk aversion of member k_1 is higher than member k_2. This is demonstrated by that the red equivalence curve is lying above the blue reference line, or, more generally, above all other equivalence curves, with which member k_1 invests more than 750.

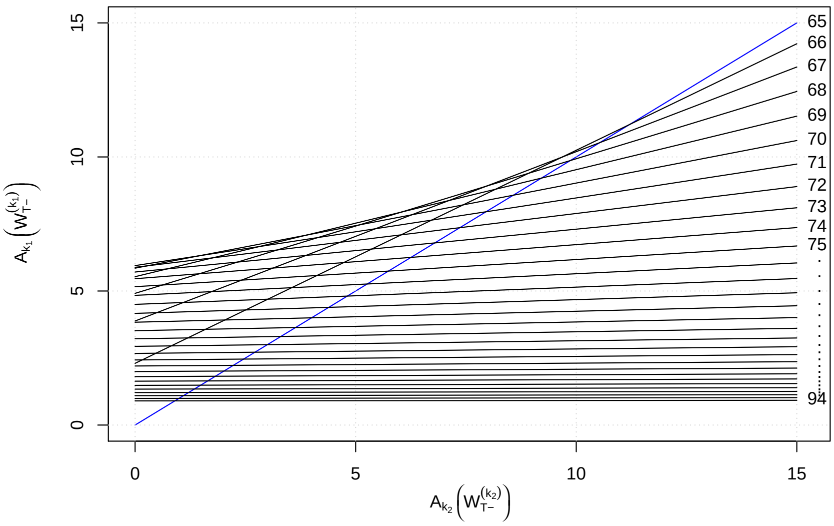
Pool Size Effect

Before ending the numerical illustration, we look at the influence of the total number of members in the fund. The portfolio of members remains the same as above, except that the number of members at each age is increased to 50, hence the fund now consists of a total of $M = 1500$ members.

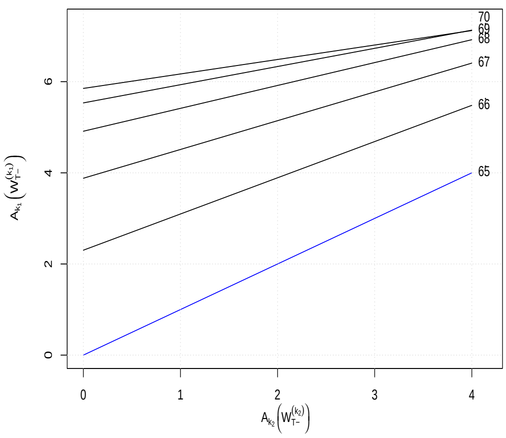
Suppose members k_1 and k_2 both invest 1 000 into the fund. Figure 3.6a is the graph of the updated equivalence curves upon fixing member k_2 to be 65 years old and varying member k_1's age. While at first glance the graph looks very similar to Figure 3.3a, pay attention to that the scale of the x- and y-axes is different. Figures 3.6b and 3.6c are a zoom-in from Figure 3.6a with various assumed ages for member k_1, the scale of the x-axis in which is set to be the same as that in Figure 3.3a. Directly comparing with Figure 3.3b and 3.3c, it is clear that now all equivalence curves are shifted upwards, except if member k_1 is equally old as member k_2. The upward shift is due to the lowered variance of the survival gain brought by a larger pool size (refer to observation A in Section 3.1). Since now the risk for all members are small, whereas the older member k_1 receives an expected gain that is higher than that for k_2, intuitively member k_1 now benefits more from the fund, unless she is much more risk averse than member k_2.

Likewise, suppose members k_1 and k_2 are both 80 years old, we fix the wealth of member k_2 to be 1 000 and vary the wealth of member k_1.

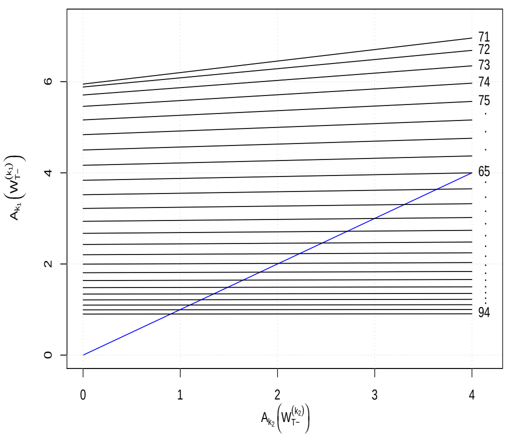
If we assume that both members benefit from the same increase in utility per



(a) Overview.



(b) Zoom-in from Figure 3.6a with member k_1 set as 65-70 years old.

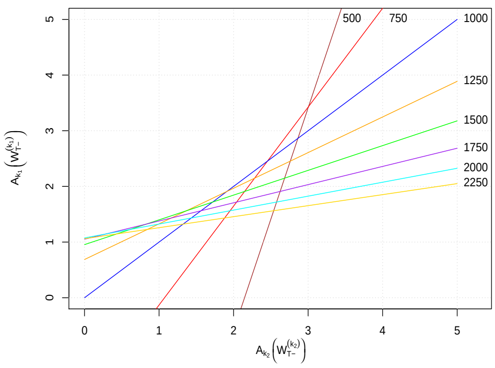


(c) Zoom-in from Figure 3.6a with member k_1 set as 71-94 years old.

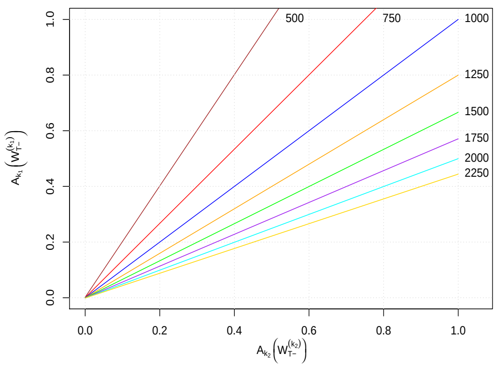
Figure 3.6: Equivalence curves for members k_1 and k_2, when the pool is enlarged to $M = 1500$ members. Assume both members invest a wealth of 1000 and member k_2 is 65 years old. The numbers on the right are the assumed age of member k_1 for the corresponding equivalence curves.

unit increase in wealth, we arrive at Figure 3.7a for the updated equivalence curves. Again, the graph has a form identical to Figure 3.4, yet see that the scale of the x- and y-axes is different. The same reasoning applies. Comparing to Figure 3.4, the equivalence curves for member k_1 investing more than 1 000 has shifted up, because member k_1 benefits from a drop in the variance of the survival gain to a greater extent than member k_2. The equivalence curves shift down in the opposite case that member k_1 invests less than member k_2.

Under the other assumption that both members benefit from the same increase in utility per percentage increase in wealth, the updated equivalence curves are as displayed in Figure 3.7b. Upon comparing, this is almost the same as Figure 3.5 and no apparent difference is observable. Recall that the slope of an equivalence curve is the ratio of the variance of survival gain faced by the two members. Here, the strong similarity between the two figures is indicating that, an enlarged pool size has nearly no effect on the ratio, equivalent to that the variances of the survival gain faced by the two members drop at a similar rate.



(a) Assume both members benefits from the same increase in utility from one unit increase in wealth.



(b) Assume both members benefits from the same increase in utility from one percentage increase in wealth.

Figure 3.7: Equivalence curves for members k_1 and k_2, when the pool is enlarged to $M = 1500$ members. Assume both are 80 years old and member k_2 invests 1 000. The numbers on the right are the assumed wealth of member k_1 for the corresponding equivalence curves.

3.4 Summary

In this chapter, we have understood the influence of a member's wealth-mortality profile on her actuarial gain obtained from the annuity overlay fund, conditional on that the concerned member survives until the terminal time. The expected change in utility for a member due to pooling risks through the annuity overlay fund is taken as the measure of the attractiveness of the fund, and a comparison on the fund's attractiveness between two members with heterogeneous wealth-mortality profiles as well as different risk aversion levels is conducted.

The contents in this chapter can be, among others, used by fund providers for forecasting the popularity of the annuity overlay fund among individuals with different wealth-mortality profiles, given that their utility functions on wealth and risk aversion levels are available. For instance, if elderlies are known to be significantly more risk averse than the young, the fund would be relatively less welcomed by the elderlies, because of the larger variance on the survival gain that elderlies face. The pool size also has a significant influence on the attractiveness of the fund for individuals with different profiles. The larger the pool is, the smaller the risk faced by all members is. Since the expected survival gain increases with age, the annuity overlay fund with a large pool of members is more attractive to the elderlies, as long as the old and the young are similarly risk averse.

The analysis in this chapter focuses on the difference in expected utility gain of fund members with heterogeneous wealth-mortality profiles, provided that their risk aversion levels are known. The correlation between the risk aversion level and the wealth-mortality profile of a member, while is crucial for applying the contents in this chapter, is out of the scope of the topic and is not studied here.

Chapter 4

Application in Retirement Aspect

In this chapter, we investigate the possibility of applying the annuity overlay fund to support retirement consumption. We are motivated to construct a way of managing the annuity overlay fund, such that the fund imitates an annuity: a stream of payment being *constant in expectation* is to be provided to fund members.

With the proposed scheme, unlike with a conventional annuity, fund members are no longer guaranteed some definite payments, but experience some level of uncertainty in the payment amount. Only in expectation do members receive a constant stream of payment as with an annuity. On the other hand, since the annuity overlay fund is simply a mechanism pooling and redistributing the wealth of members in each period, insurance companies bear theoretically no risk in running the fund but act only as a fund manager, the risk capital that companies have to bear is in turn very low. Therefore, applying such scheme in retirement aspect, this can be regarded as the insurance companies transferring the risk back to the insured (the retirees who are also the fund members), the insured benefits not from the insurer's protection but from the random actuarial gain offered by the fund.

In the following, we describe the proposed operation method in details, with which such constant expected payments can be achieved, afterwards provide methodology on the calculations and also examine the level of uncertainty of the payments.

Throughout the remainder of this chapter, each time interval is taken as one year for simplicity in illustration. We consider the case where there is no risky investment and wealth from all members grows at the same constant risk-free rate, i.e. $R_t^{(k)} = r$ for some constant $r \geq 0$, for any arbitrary member k in the fund and for any time $t = 1, 2, 3, \ldots$. The results can be easily generalized to the case where wealth from all members is invested into the same portfolio consisting also risky assets.

4.1 Fund Operation

To obtain a stream of benefit payment being constant in expectation over time, we propose the following.

Consider that upon reaching the retirement age, an individual enters the fund and invests an initial wealth amount into the fund. This amount grows with the market return throughout the year. Clearly, at the end of the year, the accumulated wealth would award the member some random survival gain upon her survival, the expected amount of which being that calculated by equation (3.1). This is directly paid out to the member as one part of the retirement benefit payment.

After concluding a round of pooling in one year, the fund starts another round of pooling for another year, where all members who have survived the past year must remain in the fund and participate in the next round of pooling. At the same time, the fund admits some new retirees into the fund to offset the drop in number of members due to the perished members in the past year, so that the pool size does not shrink relative to the past year.

Now, at the end of a year, for a surviving member, who is bound to participate in the fund again in the next year, part of her wealth invested in the fund in the past one year is to be withdrawn, so that the new amount of wealth remaining in the fund for the next year is reduced. This withdrawal is paid back to the member and constitutes the second part of the retirement benefit payment.

All in all, upon survival at each time point t where a round of pooling ends, a member receives a lump-sum comprising of two elements: the survival gain generated in the past year from the remaining wealth in fund, and the withdrawal from the fund at time t. The withdrawal amounts are set in a way, such that the two elements together form a stream of payment that is constant over time in expectation. An illustration of the proposed mechanism is sketched in Figure 4.1.

This process goes on. From the perspective of the fund issuer, each year she

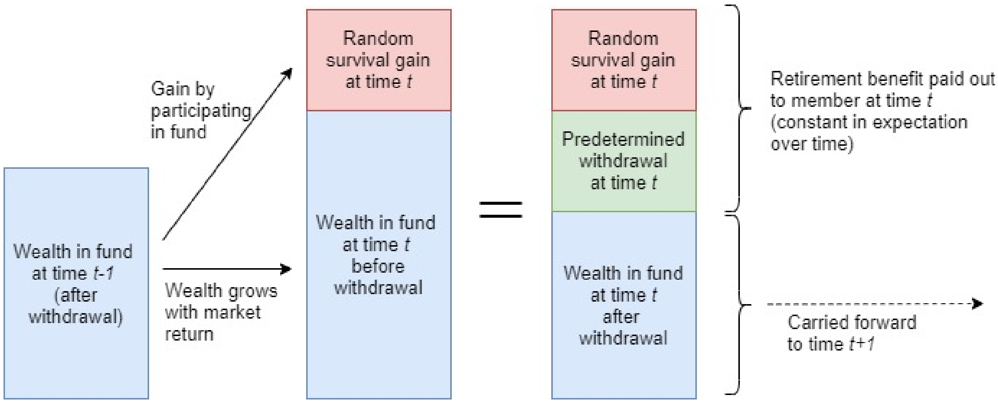


Figure 4.1: Illustration of the proposed mechanism upon survival of a member.

is hosting a new round of pooling, where she has the surviving members from the past year together with some newly entered retirees as fund members. From the perspective of a fund member, she is receiving some retirement benefit each year, which is the sum of the random survival gain and the fixed withdrawal, until the time she perishes. In case of death, the member's remaining wealth will be distributed among the pool in accordance to the allocation principle of the annuity overlay fund.[1]

The withdrawals are the core of the proposed mechanism and they are here for two reasons. First, since individuals have no bequest motives (assumed at the very beginning), at the end of the day, wealth invested into the fund should be planned to be withdrawn and consumed at some time point, so as to maximize utility from the money. Second, the withdrawals act as a control to regulate the expected survival gain, the consequential decreasing wealth in the fund counterbalances the increasing expected rate of survival gain, which makes a constant stream of payment in expectation possible. Recall that according to the principle of the annuity overlay fund, each member is assigned some percentage of her own mortality rate as the expected rate of survival gain, so if all else unchanged, the expected survival gain must increase (exponentially under most commonly used mortality models) as the member ages.

The terminologies below will be involved in this chapter:

- "Retirement benefit" refers to the periodic payment to a surviving fund

[1]This means apparently a deceased member again also receives a small actuarial gain from the period in which she passes away. However since actuarial gain upon death is not the focus of this thesis, this is continued to be left out.

member, which is the sum of the fixed withdrawal amount and the random survival gain.

- "Withdrawal strategy" refers to the set of remaining wealth amount in fund (after withdrawal) w.r.t. time, which obviously determines the withdrawal amount at each time point.

4.2 Infinite Pool

We start with the limiting case, where there are infinite members in the annuity overlay fund.

Suppose time $t = 0$ is the time when an individual reaches the retirement age $X_r > 0$ and enters the fund. At each subsequent time $t = 1, 2, 3, \ldots$, where the member will be at age X_r+t, one round of pooling completes and some retirement benefit is paid to the member upon her survival. We assume that there is a limiting age $X_{lim} > X_r$ at which no individual can live beyond, therefore the last payment time is at most at time $\tau = X_{lim} - X_r$.

Recall that with an infinite pool, one member's survival gain simply equals the product of her probability of death and wealth amount, and is independent of other members (refer to equations (3.4) and (3.5) and the discussions therein). The superscript (k) is therefore unnecessary and is dropped in the following to save us from the trouble of excessive notations.

Denote by W_0 the initial wealth invested by the individual into the fund for obtaining the later retirement benefits. Now we want to look for the remaining wealth in the fund of the member w.r.t. time t, denoted by $W_t > 0$, such that at each payment time, the withdrawal together with the expected survival gain generated by the fund from the preceding period add up to the same amount. The withdrawal is the difference between the remaining wealth at the previous payment time rolled up with market return and the remaining wealth at the current payment time. That is, denote the withdrawal amount at time t by $D_t \geq 0$, then $D_t = W_{t-1}(1 + r) - W_t$.

Mathematically, we want to look for the W_t's, $t = 1, 2, \ldots, \tau$, such that the retirement benefit $D_t + \mathbb{E}\left(G_t^{(k)} \middle| \mathcal{F}_{t-1}, N_t^{(k)} = 0\right)$ remains the same for all $t = 1, 2, \ldots, \tau$.

Here, the survival gain at time t for a member k is $\mathbb{E}\left(G_t^{(k)}\middle|\mathcal{F}_{t-1}, N_t^{(k)}=0\right) = {}_tQ_{t-1}^{(k)}W_{t-}^{(k)}$ with certainty (from equation (3.3)). In the current setting, ${}_tQ_{t-1}^{(k)}$ is the one-year death probability for the member when she is at age X_r+t-1, so ${}_tQ_{t-1}^{(k)} = q_{X_r+t-1}$, where q_X is the standard actuarial notation for the one-year death probability of an individual at age X, whereas $W_{t-}^{(k)} = W_{t-1}(1+r)$ is her wealth in the fund before pooling at time t. Our goal is then equivalent to finding the solution to the following system of linear equations:

$$\begin{aligned} &W_{t-1}(1+r) - W_t + q_{X_r+t-1}W_{t-1}(1+r) \\ =&W_t(1+r) - W_{t+1} + q_{X_r+t}W_t(1+r) \end{aligned} \tag{4.1}$$

for all $t = 1, 2, \ldots, \tau-1$.

Clearly, the solution of a member's wealth sequence W_t, $t = 0, 1, 2, \ldots, \tau$, is also independent of other members and is without randomness.

Since there will be no more payment after time τ (as the member must have perished before the next payment time), it makes sense to set $W_\tau = 0$ so that all remaining wealth of the member could be withdrawn and consumed before deceasing and thus the retirement benefit can be maximized. Now we have $\tau-1$ unknowns with $\tau-1$ equations. Rearranging terms in the system of equations (4.1), the system can be expressed in the matrix form

$$\mathbf{AW} = \mathbf{B}, \tag{4.2}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,\tau-1} \\ \vdots & \ddots & \vdots \\ a_{\tau-1,1} & \cdots & a_{\tau-1,\tau-1} \end{pmatrix}$$

with entries

$$a_{i,j} = \begin{cases} 2+r+(1+r)q_{X_r+i} & \text{for } i=j, \\ -(1+r)(1+q_{X_r+j}) & \text{for } i=j+1, \\ -1 & \text{for } i=j-1, \\ 0 & \text{otherwise }, \end{cases}$$

$$\mathbf{W} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_{\tau-1} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} W_0(1+r)(1+q_{X_r}) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The solution to $\mathbf{W}$, which is what we look for, is thus $\mathbf{A}^{-1}\mathbf{B}$.

With $\mathbf{W}$ being known, it is then straightforward to calculate the withdrawal amounts at each payment time, by $D_t = W_{t-1}(1+r) - W_t$ for all $t = 1, 2, \ldots, \tau$, and the retirement benefit that is constant over time, which is

$$D_t + \mathbb{E}\left(G_t^{(k)} \middle| \mathcal{F}_{t-1}, N_t^{(k)} = 0\right) = W_{t-1}(1+r) - W_t + q_{X_r+t-1} W_{t-1}(1+r)$$

with any $t = 1, 2, \ldots, \tau$.

4.3 Finite Pool

We proceed to the more complicated case, where the pool size is finite. Since in this case, each member's expected survival gain depends on all other members' wealth-mortality profile as well as the pool size, we must fix some restrictions on managing the pool, so that the expected survival gain is controllable. In the following, we first list the assumptions on members' profiles an the requirements on the fund management, then provide methodology to solve for the withdrawal strategy under these settings.

4.3.1 Assumptions and Fund Management

Denote again by $X_r > 0$ the retirement age.

In order that a solution exists in the case of a finite pool, we assume the following:

1. There exists a limiting age $X_{lim} > X_r$, at which no individual can live beyond.
2. The one-year death probability w.r.t. age remains unchanged throughout time.

Denote by $\tau := X_{lim} - X_r$. The fund is to be managed in accordance to the following:

1. The fund is allowed for members at age in $\{X_r, X_r + 1, \ldots, X_{lim} - 1\}$.
2. At the time the fund is established, the actual age distribution of members follows the theoretical survival probability. That is, denote the number of members at the retirement age X_r by $C \in \mathbb{N}$, then the number of members at age $X_r + t$ is close to $C \cdot {}_tp_{X_r}$ for all $t = 1, 2, \cdots, \tau - 1$, where ${}_tp_X$ is the standard actuarial notation denoting the probability of an individual aged X surviving the next t years. Further, members at each age invest an amount according to the withdrawal strategy obtained from the upcoming calculations.
3. At the beginning of each subsequent period, there is the same number of C new members at age X_r joining into the fund due to retiring, and members who attain the limiting age X_{lim} leave the fund automatically.
4. Free exit is prohibited, members leave the fund either because of death or attaining the limiting age.
5. Upon entrance, new members decide an initial investment amount among $Z \in \mathbb{N}$ choices of amounts permitted by the fund manager, as long as the portion of new members investing in each amount remains the same in each period. We term this set of choices as the "set of permissible initial wealth amount" in the sequel.
6. All members follow the withdrawal strategy laid down.

With the assumptions and management regulations, the portfolio of fund members can retain a similar form throughout time. This is possible particularly because of point 3 above, that there are constantly the same amount of new members at the same age entering the fund, and point 6, that all members are restricted to act in some predetermined way.

4.3.2 Calculations

Consider for the time being the simple case, where the set of permissible initial wealth amount is the singleton $\{W_0 > 0\}$, so all new retirees entering the fund invest the same initial wealth W_0 into the fund.

First note that, what is random in the setting is the number of surviving members at each age in the fund. Given the assumption that the mortality rate w.r.t. age does not change throughout time, the distribution of the number of members at age $X_r + t, t = 1, 2, \ldots, \tau - 1$, remains unchanged at any time. Therefore, the random number of members at age $X_r + t$ at any time can be denoted by the same random variable L_t. This further implies, given the withdrawal strategy fixed, the total randomness experienced by any cohort at any time stays the same. Consequently, for all cohorts the withdrawal strategy is also just the same.

The fund can now be considered simply as the following. At any time, there exists τ groups in the fund, with each group being the members at age $X_r + t$ and investing some wealth amount $W_t > 0$ in the fund, $t = 0, 1, \ldots, \tau - 1$. The number of members at age $X_r + t$ is L_t, with $L_0 = C$ and L_t random for $t = 1, 2, \ldots, \tau - 1$. The one-year mortality rate for members at age $X_r + t$ is q_{X_r+t}.

Same as in the case of an infinite pool, W_τ should naturally be zero, meaning all remaining wealth in the fund is withdrawn upon the member surviving until age X_{lim} and leaving the fund automatically. Our problem now is to solve for the set of W_t's, $t = 1, 2, \ldots, \tau - 1$, such that the withdrawal amount together with the random survival gain is constant in expectation. For cleaner expression, the vector $(W_1, W_2, \ldots, W_{\tau-1})$, which is what we seek, is denoted by $\mathbf{W}$ throughout the remaining calculations.

Proposition 4.1 *For a finite pool, where there are C new members at age X_r entering the fund at the beginning of every period and every new member invests W_0 upon entrance, the solution to the withdrawal strategy is $\mathbf{W}$, with which*

$$\begin{aligned} &W_{t-1}(1+r) - W_t + q_{X_r+t-1}W_{t-1}(1+r)\left[1 - q_{X_r+t-1}W_{t-1}\Theta(\mathbf{W})\right] \\ =&W_t(1+r) - W_{t+1} + q_{X_r+t}W_t(1+r)\left[1 - q_{X_r+t}W_t\Theta(\mathbf{W})\right] \end{aligned} \tag{4.3}$$

is satisfied for all $t \in [1, 2, \ldots, \tau - 1]$, where $W_\tau = 0$ and

$$\begin{aligned} &\Theta(\mathbf{W}) \\ =& \sum_{\substack{L_1, L_2, \ldots, L_{\tau-1} \\ \in \{0,1,\ldots,C\}}} \frac{1}{\sum_{n=0}^{\tau-1} q_{X_r+n} W_n L_n} \prod_{n=1}^{\tau-1} \left[\binom{C}{L_n} ({}_n p_{X_r})^{L_n} (1 - {}_n p_{X_r})^{C-L_n} \right]. \end{aligned} \tag{4.4}$$

Proof. The probability that a member survives until age $X_r + t$ given that she enters the fund, i.e. surviving at age X_r, is ${}_tp_{X_r}$. Hence, each member's survival status after entering the fund for a period of time t follows the distribution $Ber({}_tp_{X_r})$ with 1 being surviving and 0 otherwise. As members' survival processes are independent of each other's (already assumed in the set up in Donnelly et al. (2014)), we have $L_t \sim Bin(C, {}_tp_{X_r})$ and L_t's are independent of each other for all $t = 1, 2, \ldots, \tau - 1$.

Denote by G_t^* the random survival gain for a member at age $X_r + t$ for all $t = 1, 2, \ldots, \tau$, the expectation of which is

$$\begin{aligned}
\mathbb{E}\left[G_t^*\right] =& \mathbb{E}\left[\mathbb{E}\left[G_t^* \middle| L_1, L_2, \ldots, L_{\tau-1}\right]\right] \\
=& \mathbb{E}\left[q_{X_r+t-1}W_{t-}\left(1 - \frac{q_{X_r+t-1}W_{t-}}{\sum_{n=1}^{\tau} q_{X_r+n-1}W_{n-}L_{n-1}}\right)\right] \\
=& \mathbb{E}\left[q_{X_r+t-1}W_{t-1}(1+r)\left(1 - \frac{q_{X_r+t-1}W_{t-1}(1+r)}{\sum_{n=1}^{\tau} q_{X_r+n-1}W_{n-1}(1+r)L_{n-1}}\right)\right] \\
=& q_{X_r+t-1}W_{t-1}(1+r)\left[1 - q_{X_r+t-1}W_t \cdot \mathbb{E}\left[\frac{1}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n}\right]\right], \qquad (4.5)
\end{aligned}$$

where the first equality follows from equation (3.1).

Further denote by $f_{L_1,L_2,\ldots,L_{\tau-1}}(L_1, L_2, \ldots, L_{\tau-1})$ the joint probability mass function of $L_1, L_2, \ldots, L_{\tau-1}$, $f_{L_t}(L_t)$ the probability mass function of L_t for all $t = 1, 2, \ldots, \tau - 1$, and $\Theta(\mathbf{W}) := \mathbb{E}\left[\frac{1}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n}\right]$. Then

$$\begin{aligned}
&\Theta(\mathbf{W}) \\
=& \sum_{\substack{L_1,L_2,\ldots,L_{\tau-1} \\ \in\{0,1,\ldots,C\}}} \frac{1}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n} \cdot f_{L_1,L_2,\ldots,L_{\tau-1}}(L_1, L_2, \ldots, L_{\tau-1}) \\
=& \sum_{\substack{L_1,L_2,\ldots,L_{\tau-1} \\ \in\{0,1,\ldots,C\}}} \frac{1}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n} \cdot \prod_{n=1}^{\tau-1} f_{L_n}(L_n), \qquad (4.6)
\end{aligned}$$

substituting the probability mass function of binomial distribution

$$f_{L_t}(L_t) = \binom{C}{L_t}({}_tp_{X_r})^{L_t}(1 - {}_tp_{X_r})^{C-L_t}$$

for all $t = 1, 2, \ldots, \tau - 1$ gives equation (4.4).

Now we want to solve for the set of remaining wealth in fund $\mathbf{W}$, such that at each time point t the sum of withdrawals $D_t = W_{t-1}(1+r) - W_t$ and the expected survival gain is the same, that is,

$$W_{t-1}(1+r) - W_t + \mathbb{E}[G_t^*] = W_t(1+r) - W_{t+1} + \mathbb{E}[G_{t+1}^*] \tag{4.7}$$

holds for all $t = 1, 2, \ldots, \tau-1$. the claim follows directly by substituting equations (4.5) and (4.6) into equation (4.7). □

When $\mathbf{W}$ is solved, the variance of the retirement benefit w.r.t. age, which is simply the variance of the survival gain, can be derived.

Proposition 4.2 *For a finite pool, where there are C new members at age X_r entering the fund at the beginning of every period and every new member invests W_0 upon entrance, given the withdrawal strategy $\mathbf{W}$, the variance of the retirement benefit received by a member at age $X_r + t + 1$ is*

$$\left[q_{X_r+t} W_t (1+r)\right]^2 \left[\Phi\left(\mathbf{W}\right) - \left(q_{X_r+t} W_t \Theta\left(\mathbf{W}\right)\right)^2 + q_{X_r+t} W_t^2 \left(q_{X_r+t} - 1\right) \Psi\left(\mathbf{W}\right)\right], \tag{4.8}$$

where $\Theta\left(\mathbf{W}\right)$ is defined as in equation (4.4),

$$\Phi\left(\mathbf{W}\right) = \sum_{\substack{L_1, L_2, \ldots, L_{\tau-1} \\ \in \{0,1,\ldots,C\}}} \frac{\sum_{n=0}^{\tau-1} q_{X_r+n} W_n^2 L_n}{\left(\sum_{n=0}^{\tau-1} q_{X_r+n} W_n L_n\right)^2} \prod_{n=1}^{\tau-1} \left[\binom{C}{L_n} \left({}_n p_{X_r}\right)^{L_n} \left(1 - {}_n p_{X_r}\right)^{C-L_n}\right]. \tag{4.9}$$

and

$$\Psi\left(\mathbf{W}\right) = \sum_{\substack{L_1, L_2, \ldots, L_{\tau-1} \\ \in \{0,1,\ldots,C\}}} \frac{1}{\left(\sum_{n=0}^{\tau-1} q_{X_r+n} W_n L_n\right)^2} \prod_{n=1}^{\tau-1} \left[\binom{C}{L_n} \left({}_n p_{X_r}\right)^{L_n} \left(1 - {}_n p_{X_r}\right)^{C-L_n}\right]. \tag{4.10}$$

Proof. From equations (3.1) and (3.2),

$$\begin{aligned}
&\operatorname{Var}\left[G_{t+1}\middle|L_1, L_2, \ldots, L_{\tau-1}\right] \\
=&\left(\frac{q_{X_r+t}W_t(1+r)}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_n(1+r)L_n}\right)^2 \\
&\cdot\left(\sum_{n=0}^{\tau-1} q_{X_r+n}\left(W_n(1+r)\right)^2 L_n - q_{X_r+t}\left(W_t(1+r)\right)^2\right) \\
=&\left(\frac{q_{X_r+t}W_t}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n}\right)^2\left(\sum_{n=0}^{\tau-1} q_{X_r+n}\left(W_n(1+r)\right)^2 L_n - q_{X_r+t}\left(W_t(1+r)\right)^2\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[G_{t+1}\middle|L_1, L_2, \ldots, L_{\tau-1}\right] \\
=&q_{X_r+t}W_t(1+r)\left(1-\frac{q_{X_r+t}W_t(1+r)}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_n(1+r)L_n}\right) \\
=&q_{X_r+t}W_t(1+r)\left(1-\frac{q_{X_r+t}W_t}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n}\right).
\end{aligned}$$

For all $t = 1, 2, \ldots, \tau$, the variance of the survival gain for a member at age $X_r + t + 1$ is

$$\begin{aligned}
&\operatorname{Var}\left[G^*_{t+1}\right] \\
=&\mathbb{E}\left[\operatorname{Var}\left[G_{t+1}\middle|L_1, L_2, \ldots, L_{\tau-1}\right]\right] + \operatorname{Var}\left[\mathbb{E}\left[G_{t+1}\middle|L_1, L_2, \ldots, L_{\tau-1}\right]\right] \\
=&\left[q_{X_r+t}W_t(1+r)\right]^2\mathbb{E}\left[\frac{\sum_{n=0}^{\tau-1} q_{X_r+n}W_n^2L_n}{\left(\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n\right)^2}\right] \\
&-q_{X_r+t}^3W_t^4(1+r)^2\mathbb{E}\left[\frac{1}{\left(\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n\right)^2}\right] \\
&+\left(q_{X_r+t}W_t\right)^4(1+r)^2\left[\mathbb{E}\left[\frac{1}{\left(\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n\right)^2}\right]-\mathbb{E}\left[\frac{1}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n}\right]^2\right] \\
=&\left[q_{X_r+t}W_t(1+r)\right]^2\cdot \\
&\left\{\mathbb{E}\left[\frac{\sum_{n=0}^{\tau-1} q_{X_r+n}W_n^2L_n}{\left(\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n\right)^2}\right]-\left(q_{X_r+t}W_t\right)^2\mathbb{E}\left[\frac{1}{\sum_{n=0}^{\tau-1} q_{X_r+n}W_nL_n}\right]^2\right.
\end{aligned}$$

$$+ q_{X_r+t} W_t^2 (q_{X_r+t} - 1) \mathbb{E}\left[\frac{1}{(\sum_{n=0}^{\tau-1} q_{X_r+n} W_n L_n)^2}\right]\Bigg\}, \tag{4.11}$$

where the second equality follows from equations (3.1) and (3.2).

Denote $\mathbb{E}\left[\frac{\sum_{n=0}^{\tau-1} q_{X_r+n} W_n^2 L_n}{\left(\sum_{n=0}^{\tau-1} q_{X_r+n} W_n L_n\right)^2}\right] =: \Phi(\mathbf{W})$ and $\mathbb{E}\left[\frac{1}{\left(\sum_{n=0}^{\tau-1} q_{X_r+n} W_n L_n\right)^2}\right] =: \Psi(\mathbf{W})$, the expressions of $\Phi(\mathbf{W})$ and $\Psi(\mathbf{W})$ are derived straightforwardly by following the same lines as in the derivation of $\Theta(\mathbf{W})$ (refer to the derivation of equation (4.6)), from which we obtain equations (4.9) and (4.10).

Substituting equations (4.6), (4.9) and (4.10) into equation (4.11) completes the proof. □

Unfortunately, observe that direct computation of $\Theta(\mathbf{W})$, $\Phi(\mathbf{W})$ and $\Psi(\mathbf{W})$ is challenging, if not infeasible, with reasonable τ and C. All three require calculating and summing $(C+1)^{\tau-1}$ terms, an amount that quickly explodes. (τ is the number of payment times throughout the retirement duration, even considering a simple yearly payment, τ would typically be of magnitude of tens, whereas a sensible C would naturally be at least tens to hundreds.) In particular, our goal to solve for $\mathbf{W}$ from equation (4.3), which is also embedded in $\Theta(\mathbf{W})$, is impracticable.

To get over this complication, approximations for $\Theta(\mathbf{W})$, $\Phi(\mathbf{W})$ and $\Psi(\mathbf{W})$ are proposed.

Proposition 4.3 $\Theta(\mathbf{W})$, $\Phi(\mathbf{W})$ *and* $\Psi(\mathbf{W})$ *as defined respectively in equations (4.4), (4.9) and (4.10) can be approximated by*

$$\Theta(\mathbf{W}) \approx \frac{1}{\mu_1} + \frac{\sigma_1^2}{\mu_1^3} =: \theta(\mathbf{W}), \tag{4.12}$$

$$\Phi(\mathbf{W}) \approx \frac{\mu_2}{\mu_1^2} + 3\frac{\mu_2\sigma_1^2}{\mu_1^4} - 4\frac{\sigma_{12}}{\mu_1^3} =: \phi(\mathbf{W}), \text{ and} \tag{4.13}$$

$$\Psi(\mathbf{W}) \approx \frac{1}{\mu_1^2} + 3\frac{\sigma_1^2}{\mu_1^4} =: \psi(\mathbf{W}), \tag{4.14}$$

where

$$\mu_1 = C\sum_{n=0}^{\tau-1} q_{X_r+n} W_n \left({}_n p_{X_r}\right),$$
$$\mu_2 = C\sum_{n=0}^{\tau-1} q_{X_r+n} W_n^2 \left({}_n p_{X_r}\right),$$
$$\sigma_1^2 = C\sum_{n=1}^{\tau-1} \left(q_{X_r+n} W_n\right)^2 \left({}_n p_{X_r}\right)\left(1 - {}_n p_{X_r}\right),$$
$$\sigma_2^2 = C\sum_{n=1}^{\tau-1} \left(q_{X_r+n} W_{n-1}^2\right)^2 \left({}_n p_{X_r}\right)\left(1 - {}_n p_{X_r}\right),$$
$$\sigma_{12} = C\sum_{n=1}^{\tau-1} q_{X_r+n}^2 W_{n-1}^3 \left({}_n p_{X_r}\right)\left(1 - {}_n p_{X_r}\right).$$

Proof. We give a proof for $\Theta(\mathbf{W}) \approx \theta(\mathbf{W})$.

Denote by $Y_1 := \sum_{n=0}^{\tau-1} q_{X_r+n} W_n L_n$. Obviously, $\mathbb{E}(Y_1) = \sum_{n=0}^{\tau-1} q_{X_r+n} W_n \mathbb{E}(L_n) = \mu_1$ and $\mathrm{Var}(Y_1) = \sum_{n=0}^{\tau-1} \left(q_{X_r+n} W_n\right)^2 \mathrm{Var}(L_n) = \sigma_1^2$. (For discussion on distributions of L_t's refer back to the proof of Proposition 4.1.)

Define the function $g(x) = \frac{1}{x}$. Denote by $g'(x)$ and $g''(x)$ respectively the first and second order derivative of $g(x)$, so $g'(x) = \frac{-1}{x^2}$ and $g''(x) = \frac{2}{x^3}$.

Apply Taylor approximation on $g(Y_1)$ around $\mathbb{E}(Y_1)$ up to second order,[2]

$$\begin{aligned} g(Y_1) &\approx g(\mathbb{E}(Y_1)) + (Y_1 - \mathbb{E}(Y_1))g'(\mathbb{E}(Y_1)) + \frac{1}{2}(Y_1 - \mathbb{E}(Y_1))^2 g''(\mathbb{E}(Y_1)) \\ \mathbb{E}(g(Y_1)) &\approx g(\mathbb{E}(Y_1)) + \mathbb{E}\left[Y_1 - \mathbb{E}(Y_1)\right] g'(\mathbb{E}(Y_1)) + \frac{1}{2}\mathbb{E}\left[(Y_1 - \mathbb{E}(Y_1))^2\right] g''(\mathbb{E}(Y_1)) \\ &= g(\mathbb{E}(Y_1)) + \frac{1}{2}\mathrm{Var}(Y_1) g''(\mathbb{E}(Y_1)). \end{aligned}$$

Therefore,

$$\Theta(\mathbf{W}) \approx \frac{1}{\mathbb{E}(Y_1)} + \frac{\mathrm{Var}(Y_1)}{\mathbb{E}(Y_1)^3}.$$

[2] Expansion up to third order can also be done straightforwardly, when higher accuracy is desired. (For this, one could make use of the following properties: cumulants are additive for independent variables, and the third cumulant is homogeneous of degree 3.) Numerical trials have however shown that, the difference in results between expansion up to second order and third order is negligible even with very small C, and therefore only expansion to second order is demonstrated here.

Substituting $\mathbb{E}(Y_1) = \mu_1$ and $\text{Var}(Y_1) = \sigma_1^2$ yields approximation (4.12).

Approximations (4.13) and (4.14) are derived likewise. For $\Psi(\mathbf{W})$, let function $g(x) = \frac{1}{x^2}$. For $\Phi(\mathbf{W})$, let function $g(x, y) = \frac{y}{x^2}$ and apply Taylor expansion in two variables. The relevant terms are derived from the following.

Denote by $Y_2 := \sum_{n=0}^{\tau-1} q_{X_r+n} W_n^2 L_n$. $\mathbb{E}(Y_2) = \mu_2$ and $\text{Var}(Y_2) = \sigma_2^2$ are clear. Also, $\text{Cov}(Y_1, Y_2) = \sum_{n=0}^{\tau-1} q_{X_r+n}^2 W_n^3 \text{Var}(L_n) = \sigma_{12}$. □

Using Proposition 4.3, we arrive at the following:

Proposition 4.4 *For a finite pool, where there are C new members at age X_r entering the fund at the beginning of every period and every new member invests W_0 upon entrance, the withdrawal strategy can be* approximated *by the vector* $\mathbf{W}$*, with which*

$$\begin{aligned} & W_{t-1}(1+r) - W_t + q_{X_r+t-1} W_{t-1}(1+r) \left[1 - q_{X_r+t-1} W_{t-1} \theta(\mathbf{W})\right] \\ =& W_t(1+r) - W_{t+1} + q_{X_r+t} W_t(1+r) \left[1 - q_{X_r+t} W_t \theta(\mathbf{W})\right] \end{aligned} \tag{4.15}$$

is satisfied for all $t = 1, 2, \ldots, \tau - 1$, where $W_\tau = 0$ and $\theta(\mathbf{W})$ is defined as in equation (4.12).

Proof. Substitute the approximation $\Theta(\mathbf{W}) \approx \theta(\mathbf{W})$ into equation (4.3). □

While equation (4.15) is not linear in W_t's and there is no analytical solution available, it can be solved easily by numerical methods.

With $\mathbf{W}$ now at hand, the variance of the retirement benefit can be approximated.

Proposition 4.5 *For a finite pool, where there are C new members at age X_r entering the fund at the beginning of every period and every new member invests W_0 upon entrance, given the withdrawal strategy $\mathbf{W}$, the variance of the retirement benefit received by a member at age $X_r + t$ can be* approximated *by*

$$\left(q_{X_r+t} W_t(1+r)\right)^2 \left[\phi(\mathbf{W}) - \left(q_{X_r+t} W_t \theta(\mathbf{W})\right)^2 + q_{X_r+t} W_t^2 \left(q_{X_r+t} - 1\right) \psi(\mathbf{W})\right], \tag{4.16}$$

where $\theta(\mathbf{W})$, $\phi(\mathbf{W})$ and $\psi(\mathbf{W})$ are defined as in Proposition 4.3.

Proof. Substitute the approximations from Proposition 4.3 into equation (4.8).□

When the set of permissible initial wealth amount has more than one but finite elements, similar calculations can also be done as with Proposition 4.4. In this case, an analogue to equation (4.15) has to be satisfied.

Proposition 4.6 *Suppose the set of permissible initial wealth amount is* $\{W_{0,1}, W_{0,2}, \ldots, W_{0,Z}\}$*, i.e., new members can invest one of the* $Z \in \mathbb{N}$ *amounts upon retirement. At the beginning of every period,* C *new members at age* X_r *enter the fund, with some constant* $C_z \in \mathbb{N}$ *members investing the amount* $W_{0,z}$ *upon entrance for all* $z \in \{1, 2, \ldots, Z\}$ *and* $\sum_{z=1}^{Z} C_z = C$*. Then, the withdrawal strategy can be* approximated *by the set of* $W_{t,z}$*'s,* $t \in \{1, 2, \ldots, \tau - 1\}$ *and* $z \in \{1, 2, \ldots, Z\}$*, with which*

$$\begin{aligned}
&W_{t-1,z}(1+r) - W_{t,z} \\
&\quad + q_{X_r+t-1} W_{t-1,z}(1+r)\left[1 - q_{X_r+t-1} W_{t-1,z} \vartheta\left(W_{1,1}, \ldots, W_{\tau-1,Z}\right)\right] \\
=&W_{t,z}(1+r) - W_{t+1,z} \\
&\quad + q_{X_r+t} W_{t,z}(1+r)\left[1 - q_{X_r+t} W_{t,z} \vartheta\left(W_{1,1}, \ldots, W_{\tau-1,Z}\right)\right] \qquad (4.17)
\end{aligned}$$

is satisfied for all $t \in \{1, 2, \ldots, \tau - 1\}$ *and* $z \in \{1, 2, \ldots, Z\}$*, where* $W_{\tau,z} = 0$ *for all* $z \in \{1, 2, \ldots, Z\}$ *and*

$$\vartheta\left(W_{1,1}, \ldots, W_{\tau-1,Z}\right) = \frac{1}{\mu_{(z)}} + \frac{\sigma_{(z)}^2}{\mu_{(z)}^3}, \qquad (4.18)$$

with

$$\begin{aligned}
\mu_{(z)} &= \sum_{z=1}^{Z} C_z \sum_{n=0}^{\tau-1} q_{X_r+n} W_{n,z} ({}_n p_{X_r}), \\
\sigma_{(z)}^2 &= \sum_{z=1}^{Z} C_z \sum_{n=1}^{\tau-1} \left(q_{X_r+n} W_{n,z}\right)^2 ({}_n p_{X_r})(1 - {}_n p_{X_r}).
\end{aligned}$$

Proof. Denote by $L_{t,z}$, $t \in \{0, 1, \ldots, \tau - 1\}$ and $z \in \{1, 2, \ldots, Z\}$, the number of members at age $X_r + t$ in the fund, who invested the amount $W_{0,z}$ upon their entrance. We have $L_{0,z} = C_z$ and $L_{t,z}$ random for $t \in \{1, 2, \ldots, \tau - 1\}$ and $z \in \{1, 2, \ldots, Z\}$.

Analogue to discussion in the proof of Proposition 4.1, $L_{t,z}$ follows $Bin(C_z, {}_tp_{X_r})$ for all $t \in \{1, 2, \ldots, \tau - 1\}$ and $z \in \{1, 2, \ldots, Z\}$.

Denote by $G^*_{t,z}$ the random survival gain for a member at age X_r+t, who invested $W_{0,z}$ upon entrance. For all $t \in \{0, 1, \ldots, \tau - 1\}$, the expectation of $G^*_{t,z}$ is

$$\begin{aligned}
&\mathbb{E}\left(G^*_{t,z}\right)\\
=&\mathbb{E}\left[\mathbb{E}\left[G^*_{t,z}\middle|L_1, L_2, \ldots, L_{\tau-1}\right]\right]\\
=&\mathbb{E}\left[q_{X_r+t-1}W_{t-1,z}(1+r)\left(1-\frac{q_{X_r+t-1}W_{t-1,z}(1+r)}{\sum_{z=1}^{Z}\sum_{n=0}^{\tau-1}q_{X_r+n}W_{n,z}(1+r)L_{n,z}}\right)\right]\\
=&q_{X_r+t-1}W_{t-1,z}(1+r)\left[1-q_{X_r+t-1}W_{t-1,z}\cdot\mathbb{E}\left[\frac{1}{\sum_{z=1}^{Z}\sum_{n=0}^{\tau-1}q_{X_r+n}W_{n,z}L_{n,z}}\right]\right].
\end{aligned} \tag{4.19}$$

Denote by $Y_{(z)} : \sum_{z=1}^{Z}\sum_{n=0}^{\tau-1} q_{X_r+n}W_{n,z}L_{n,z}$, then $\mathbb{E}\left(Y_{(z)}\right) = \mu_{(z)}$ and $\text{Var}\left(Y_{(z)}\right) = \sigma^2_{(z)}$. Using Taylor approximation expanding up to second order (refer to proof of Proposition 4.3 for details), we obtain

$$\mathbb{E}\left[\frac{1}{\sum_{z=1}^{Z}\sum_{n=0}^{\tau-1}q_{X_r+n}W_{n,z}L_{n,z}}\right] \approx \vartheta\left(W_{1,1}, \ldots, W_{\tau-1,Z}\right), \tag{4.20}$$

where $\vartheta\left(W_{1,1}, \ldots, W_{\tau-1,Z}\right)$ is defined as in equation (4.18). Now we want to solve for the set of remaining wealth in fund w.r.t. age for each of the Z groups, that is, the $W_{t,z}$'s for all $t \in \{1, 2, \ldots, \tau - 1\}$ and $z \in \{1, 2, \ldots, Z\}$, such that

$$W_{t-1,z}(1+r) - W_{t,z} + \mathbb{E}[G^*_{t,z}] = W_t(1+r) - W_{t+1} + \mathbb{E}[G^*_{t+1,z}] \tag{4.21}$$

holds for all $t \in \{1, 2, \ldots, \tau - 1\}$ and $z \in \{1, 2, \ldots, Z\}$. Substituting approximation (4.20) and equation (4.19) into equation (4.21) gives the proposition. □

Observe that Proposition 4.6 suggests a system of $\tau - 1 \times Z$ non-linear equations with the same number of unknowns, which can again be solved numerically.

The variance of the retirement benefit for each of the Z groups under the setting in Proposition 4.6 can be easily derived by extending Propositions 4.2 and 4.3 likewise, which we leave out here.

4.4 Numerical Illustrations

We continue to take Germany as our illustrative country. The retirement age is set as $X_r = 65$, which is the official retirement age in the country until 2011[3], and the limiting age is $X_{lim} = 110$, which is the oldest age with data available on the HMD, hence $\tau = 110 - 65 = 45$. The constant risk-free rate is fixed at $r = 2\%$. The initial investment amount is $W_0 = 500000$ unless otherwise specified. For the mortality model we continue to employ the Gompertz model parametrized in Section 3.3.1.

4.4.1 Infinite Pool

With an infinite pool, the yearly constant benefit under the constructed withdrawal strategy given $W_0 = 500000$ is 31464.91.

The evolutions of the wealth in fund and the withdrawal amount are the primary focus. Equation (4.2) is solved and the two quantities are depicted respectively in Figure 4.2 and Figure 4.3.

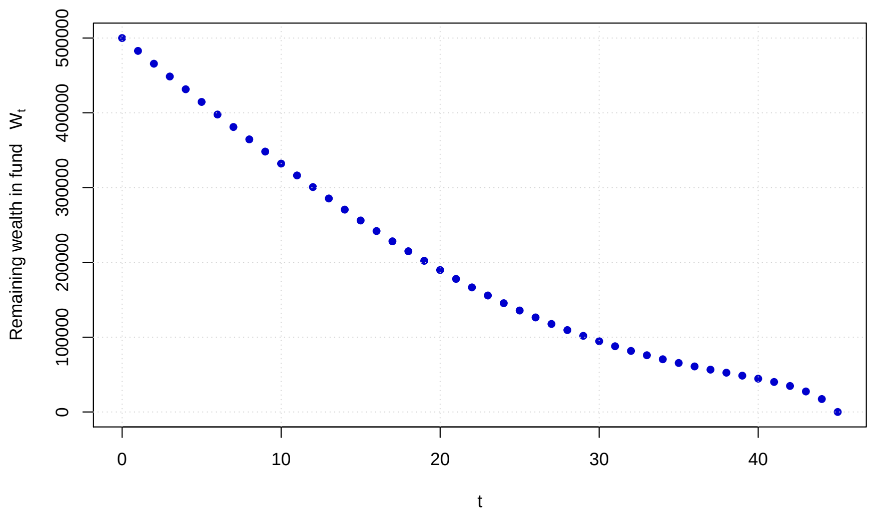


Figure 4.2: Evolution of remaining wealth in fund, $W_0 = 500000$.

[3]The retirement age in Germany has been gradually increasing since 2012 until reaching 67 in 2029. To save us from the trouble of a fractional retirement age we adopt the integer retirement age before this transition period.

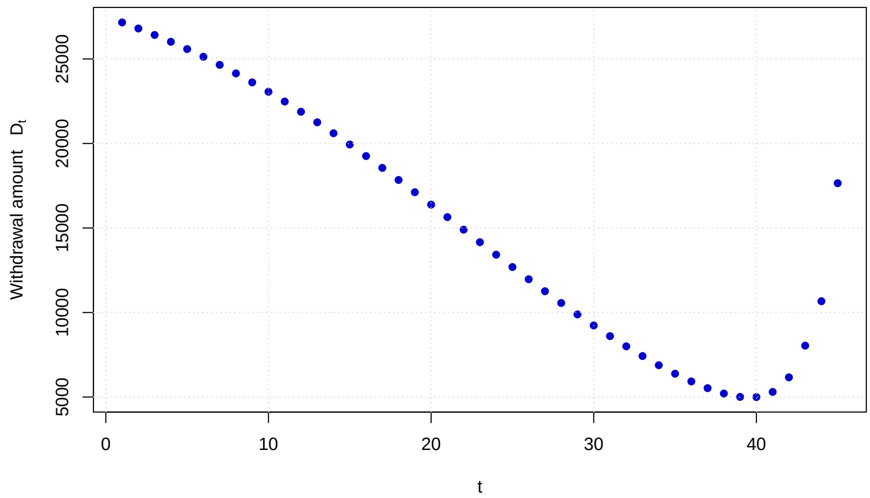


Figure 4.3: Withdrawals with respect to time, $W_0 = 500000$.

The wealth amount in fund is decreasing quite steadily, although not exactly linearly. The trend of the withdrawals has an interesting shape: it decreases steadily with age until the advanced age 105, then increases sharply. The decreasing part is easy to comprehend, as a member grows older and so her mortality rate increases, she obtains more survival gain from the fund, less withdrawal is needed to make up the same retirement benefit amount. However, at very advanced ages, the withdrawal has to be increased, because at this stage, there is little wealth remaining in the fund and the amount of survival gain attracted is limited despite the individual's high mortality rate. Increased withdrawal is needed in order that the constant benefit payment can be sustained. This can be better understood from Figure 4.4, in which the evolution of the composition of the benefit payments w.r.t. age is displayed and a drop in survival gain at the very advanced ages is shown.

A comparison between the payments from the annuity overlay fund with infinite members and that from an actuarially fair traditional annuity is made. Equation (4.2) is solved with various values of initial wealth amounts W_0, and the annual constant benefits under respective W_0's are computed. The result is sketched in Figure 4.5. On the other hand, the constant benefits that an individual would receive if the same wealth amounts are instead invested into a traditional annuity are calculated, which is done by dividing the wealth amounts by the annuity factor of an actuarially fair whole life annuity immediate offered to an insured at

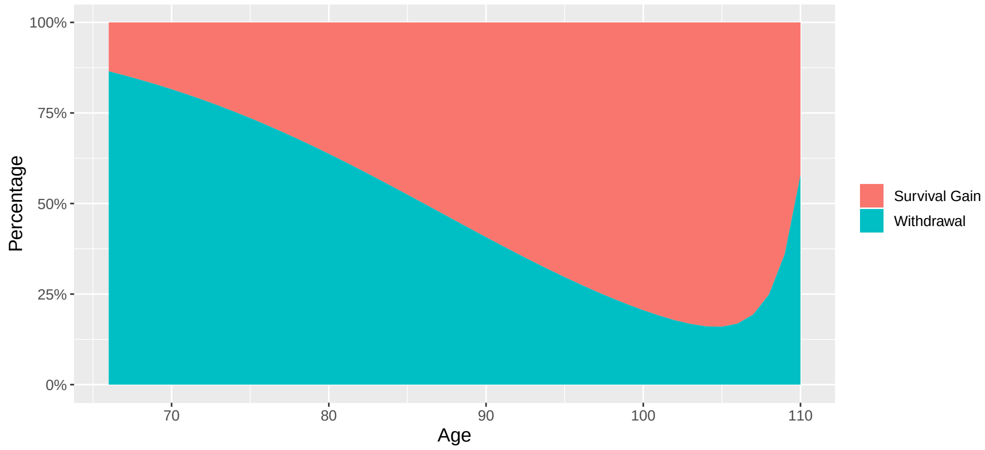


Figure 4.4: Composition of benefit payment with respect to age.

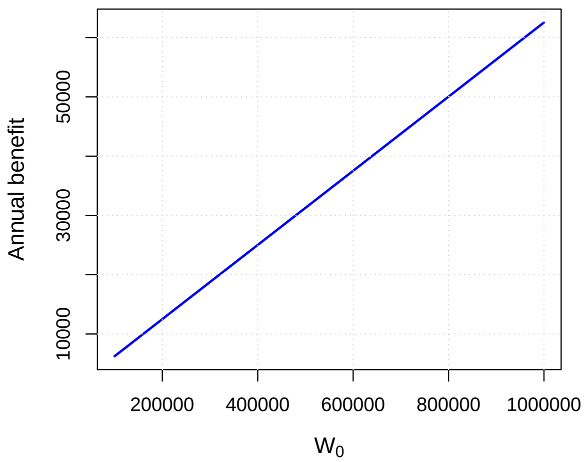


Figure 4.5: Annual retirement benefit from the annuity overlay fund with infinite members.

age $X_r = 65$. The annuity factor, denoted by a_{65}, is calculated by the formula

$$a_{65} = \sum_{t=1}^{45} (1+r)^{-t} {}_t p_{65}. \tag{4.22}$$

Under our setting $a_{65} = 15.5753$. A direct comparison confirms that, the constant annual benefit offered by the annuity overlay fund with infinite pool under the constructed withdrawal strategy offers benefit amounts very close to that an actuarially fair annuity does, with a percentage difference of -1.99% in all cases. This difference arises from the fact that, with the annuity overlay fund, there is still a small payment made to members upon their death (which we assumed that this payment brings no value to the deceased members), whereas with an annuity no payment is to be made upon death.

4.4.2 Finite Pool

With regard to a finite pool, we give numerical illustrations primarily focusing on the simple case where the set of permissible initial investment amount is the singleton $\{W_0\}$.

Equation (4.15) is solved under various values of C. The amount of the constant expected retirement benefit is given in Figure 4.6 and Table 4.1. Since the expected rate of survival gain increases with the pool size, so does the expected retirement benefit. It is found that, a pool formed by constantly allowing a relatively small number of C new retirees entering the pool already suffices to provide an expected retirement benefit that is comparable to that from an infinite pool. For instance, the expected survival gain attains 99.931% of that from an infinite pool already with $C = 25$. A level as high as 99.994% is reached when C is large equal to 300.

C	10	25	50	100	300
Expected benefit	31410.25	31443.13	31454.04	31459.48	31463.10
Percentage comparing to infinite pool	99.826%	99.931%	99.965%	99.983%	99.994%

Table 4.1: Comparison on expected benefit from finite and infinite pool.

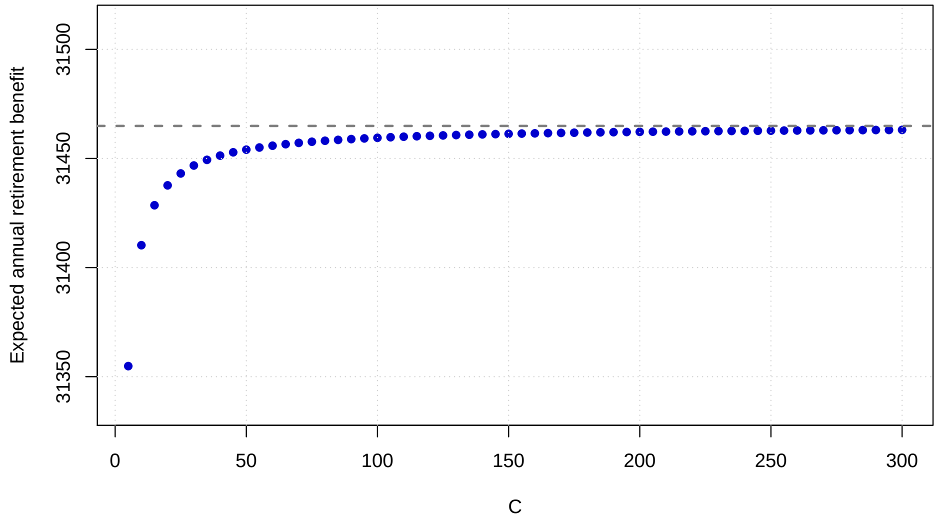


Figure 4.6: Expected annual retirement benefit in relation to C, $W_0 = 500000$. The gray dashed line indicates the benefit amount in the limiting case of an infinite pool.

The wealth in fund and the withdrawal amount w.r.t. age is compared to that with an infinite pool. The percentage differences for the two quantities are displayed respectively in Figures 4.7 and 4.8.

Referring to Figure 4.7, it is found that the remaining wealth in fund with a finite pool is always higher than that with an infinite pool. The smaller C is, corresponding to a smaller pool, the more wealth should be remained in the fund. The percentage difference is increasing with age for most of the time. Only after the very late age of 102 ($t = 37$), the percentage increase starts diminishing, until the remaining wealth finally drops to zero at the limiting age regardless of the pool size. The higher wealth amount needed in the fund is due to the fact that, comparing to an infinite pool, the expected rate of survival gain with a finite pool is always lower, more wealth has to stay in the fund to attract a comparable amount of survival gain. Meanwhile, note the small scale of the y-axis in the graph. Already with a small pool formed by $C = 10$, the percentage difference is at all time smaller than 1%. With $C = 300$ new members entering the fund at each period, the percentage difference is hardly observable from the graph.

In order that the wealth amount in fund is at all time higher, withdrawals have

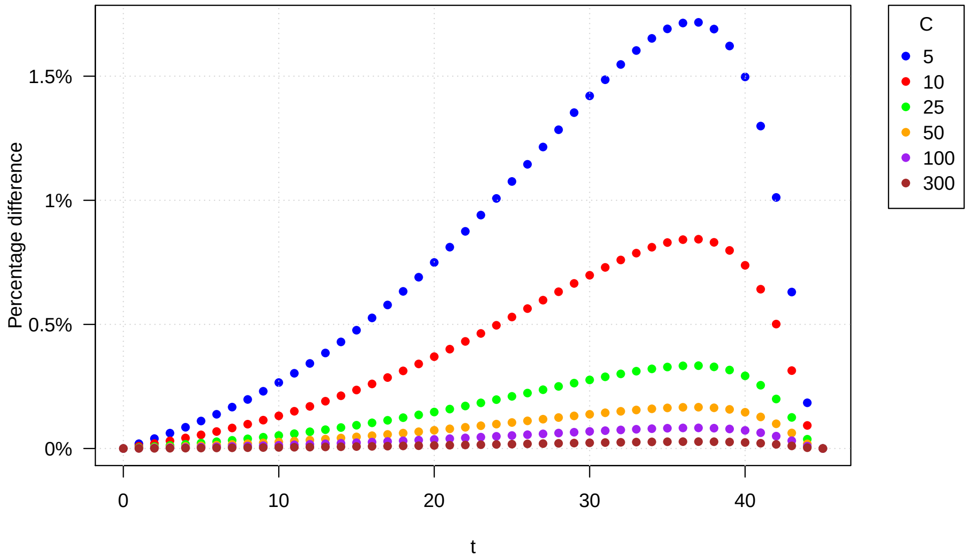


Figure 4.7: Percentage change in wealth in fund comparing to an infinite pool.

to be deferred. See Figure 4.8. The withdrawal amount is first reduced at the younger stage, and is then increased at the older stage. The smaller the pool is, the more the withdrawals should be deferred, so that more wealth remains in the fund to attract survival gain, which is seen from the larger negative percentage change at the young ages and larger positive percentage change at the advanced ages for smaller C. Relative to an infinite pool, the withdrawal amount increases with age for most of the time, corresponding to that relatively more wealth can be taken out from the fund as the expected survival gain increases. In the last few years, the percentage difference falls again. At this stage, the survival gain attracted drops due to the little wealth in fund, the portion of the retirement benefit contributed from the withdrawals must be increased again (refer again to Figure 4.3 and 4.4 and the discussion therein), the relative difference to the limiting case hence also reduces.

The coefficient of variation (CV) of the benefit payment for a member at age $X_r + t, t = 1, 2, \ldots, \tau$ is computed under various values of C with the formula

$$CV_t = \frac{\text{Standard Deviation of Survival Gain at age } X_r + t}{\text{Expected Retirement Benefit at age } X_r + t}, \tag{4.23}$$

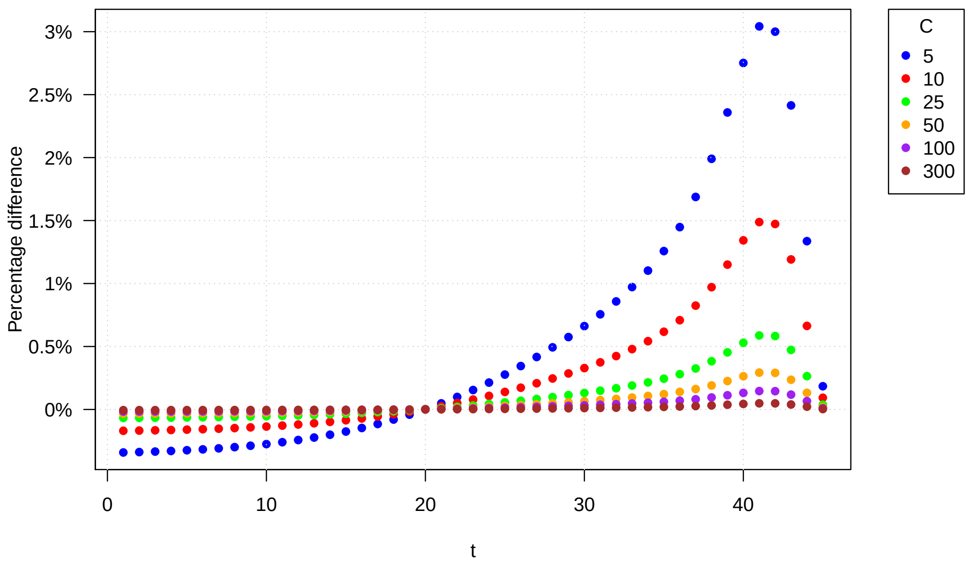


Figure 4.8: Percentage change in withdrawal amounts comparing to an infinite pool.

the result of which is given in Figure 4.9.[4] The CV is always increasing with age except after the advanced age 104 ($t = 39$). This is because the uncertainty on the retirement benefit amount only comes from the random survival gain but not from the predetermined withdrawals. Until the last several years before the limiting age, the survival gain constitutes an increasing portion of the retirement benefit as a member ages, leading to a higher level of uncertainty w.r.t. age. Yet in the last years, the portion contributed from the random survival gain drops, therefore the uncertainty on the total benefit also reduces. On the other hand, notice that C has a significant impact on the CV, especially for the old members. For instance, when C is small equal to 10, the CV for a member aged 105 could reach as high as 0.3, implying a 30% dispersion for each unit retirement benefit. The CV drops quickly as C grows from such small value. For C over 150, the CV becomes relatively stable, and for C over 250, the CV remains below 0.06 for all members.

A stochastic simulation is performed to visualize the variation in the actual retirement benefit amounts. Fixing C, the number of surviving members at

[4]The CV instead of the variance is examined, as it is here a more indicative statistic than the variance, considering that the expectation of payment is not the same for different values of C.

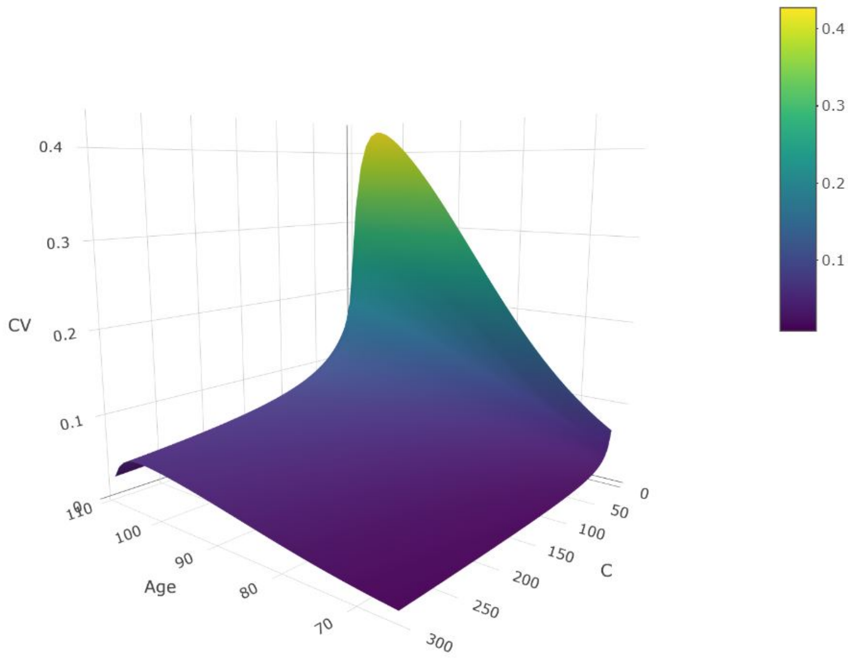
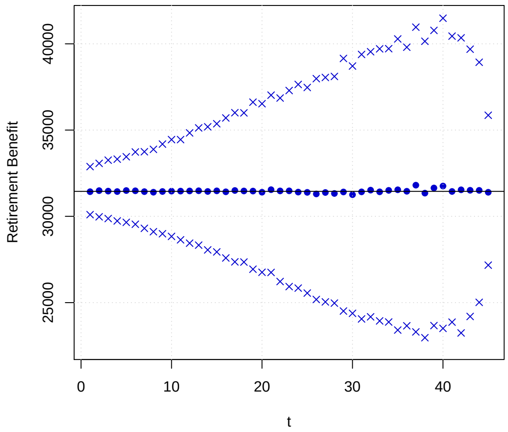


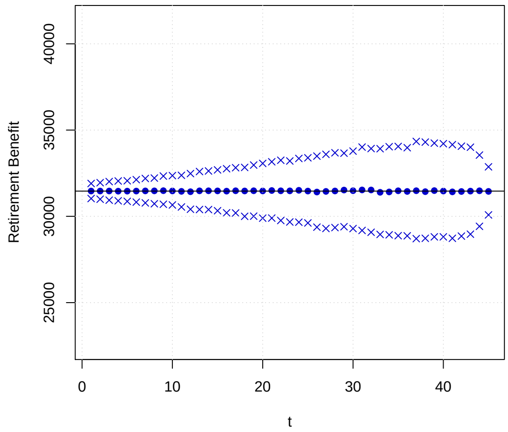
Figure 4.9: Coefficient of variation of benefit payments in relation to age and pool size.

each age at each time point is modelled according to the survival probability calculated from the fitted mortality curve. The retirement benefit amounts are then calculated under the withdrawal strategy set from the above calculation together with the modelled evolution of number of fund members. Figure 4.10 is the result when C is fixed respectively at 30 and 300, with 2000 simulations being run for each case. Obviously with a larger C, the fluctuation of the benefit amount around the expectation reduces. For $C = 30$, the maximum variation occurs at ages 103 and 104, where the 95th and 5th percentiles correspond to roughly $+30\%$ and -27% respectively from the expected value. For $C = 300$, the maximum deviation reduces to around $\pm 9\%$. As remark, the simulation result that the expected retirement benefit is almost constant w.r.t. age supports that the approximations employed (given in Proposition 4.3) are valid practically.

As reference, an example for the case where the set of permissible initial investment amount has more than one element is provided. The set is taken as $\{W_{0,1} = 500000, W_{0,2} = 250000\}$, equation (4.17) is solved under various values of C, with $C_1 = C_2 = 0.5C$. The effect on the members investing 500000 is illustrated in Figure 4.11. Now with half of the members investing less into the



(a) $C = 30$.



(b) $C = 300$.

Figure 4.10: Simulation results for the retirement benefit w.r.t. age. Circle marks are the mean payments, upper (lower) cross marks are the 95th (5th) percentile of the payments, black line is the theoretical expected value (corresponding to Figure 4.6).

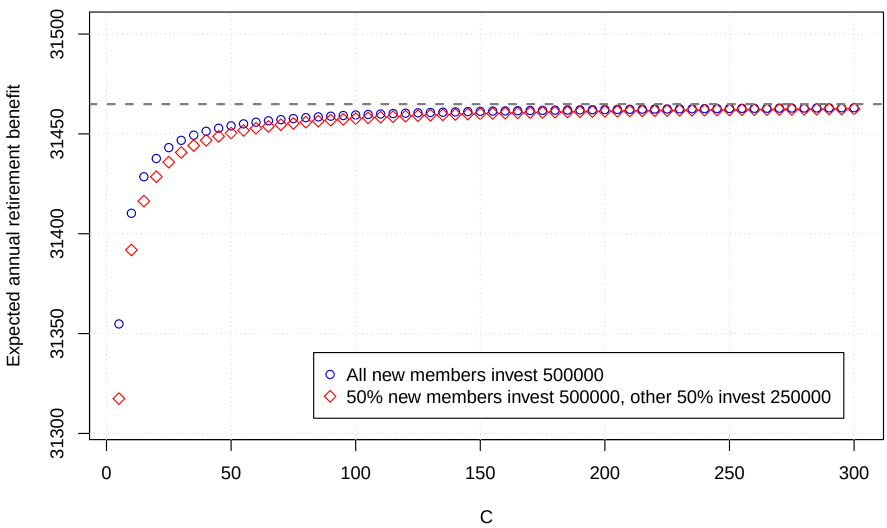


Figure 4.11: Effect on expected benefit for a member investing an initial wealth of 500000, when 50% of the members invest 50% less upon entering the fund. The gray dashed line indicates the benefit amount in the limiting case of an infinite pool.

fund upon entrance, those who invested 500000 suffer from a reduced expected retirement benefit. When C is small being below 50, the drop is obvious. As C increases and hence the pool is enlarged, the difference between the two cases diminishes. For C over roughly over 150, the effect becomes unobservable from the graph, where the expected benefit in either case is very close to that from an infinite pool. Clearly the converse holds also true, that is, for certain member, when some other members invest more into the fund, the member benefits from a higher expected retirement benefit.

4.4.3 Effect of Change in Mortality Rate

Until now, all calculations are done based on the assumption that the one-year death probability w.r.t. age does not change throughout time. In reality this assumption rarely holds. Here we investigate the effect of change in the mortality rate on the expected retirement benefit amount.

For this, a mortality curve is fitted to the German population data from 2007 and the withdrawal strategy is set according to the fitted one-year mortality rate w.r.t. age back then. Employing the same parametrization method as in Section 3.3.1, the parameters for the 2007 mortality curve are $m = 87.16$ and $b = 9.01$. Comparing with the parameters in 2017 where $m = 88.13$ and $b = 8.66$, the modal age of death has increased during the period, reflecting a longer life expectancy, and the dispersion of death time has decreased, implying individuals are passing away at a time around the modal age more concentratedly. These can be related to the phenomenons termed as "right-shift mortality" and "concentration" of the human mortality evolution (see e.g. Börger et al. (2018) for a very good summary and categorization of the various types of mortality evolution patterns).

Now suppose the population mortality level has changed from that in 2007 to the that in 2017. The effect on the expected retirement benefit from the 10-year change in mortality level is plotted in Figure 4.12. Observe that the expected retirement benefit is no longer constant in time after a change in the mortality level. The expected retirement benefit becomes lower for most of the time, since with an improved mortality level, members perish at a later age in expectation. As older members have less wealth in the fund than the younger ones, less money passes into the notional mortality account and less survival gain is awarded to the surviving members. Except at the very advanced ages, the expected retirement benefit becomes higher, this is explained by the higher mortality rate for the very advanced ages from the mortality level in 2017 than that in 2007, more survival gain is hence attracted to the very old members. Note that a larger pool size, formed by allowing more new entrants at each period, cannot remedy the distortion, as a change in the population mortality level is systemic.

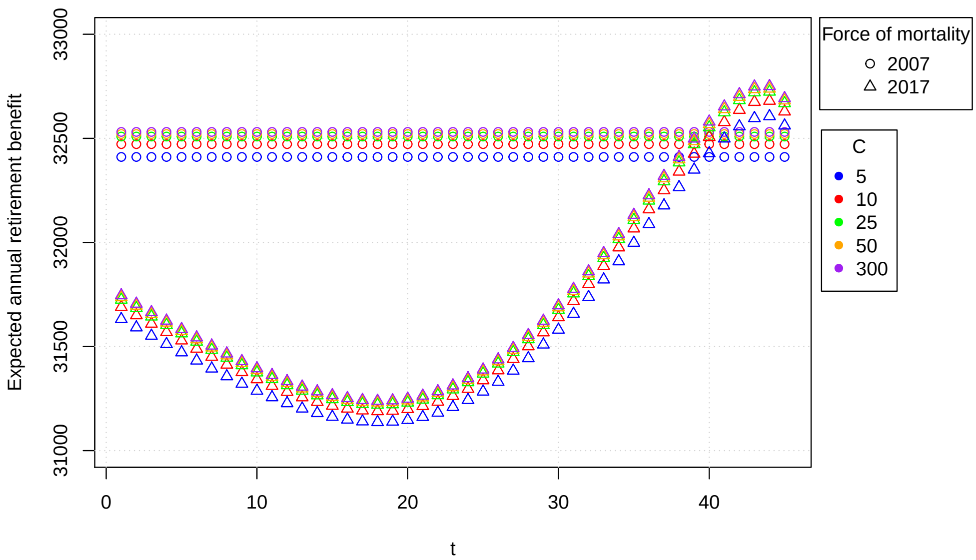
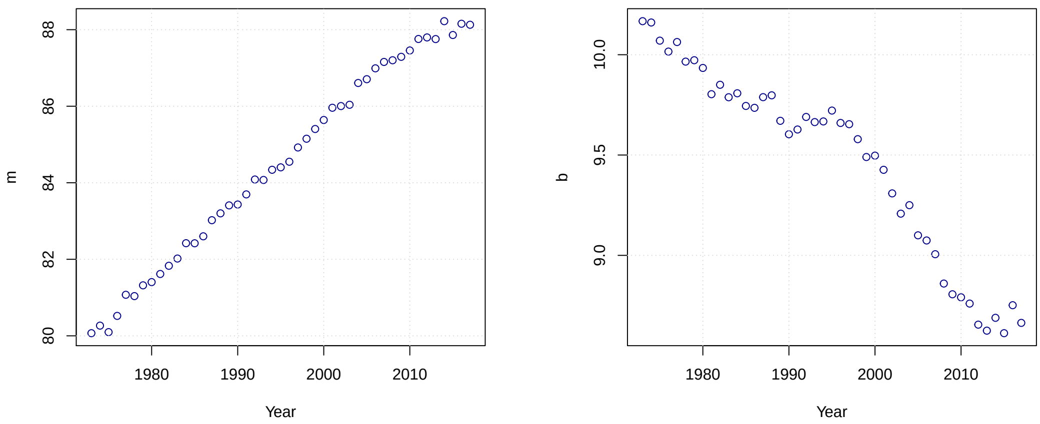


Figure 4.12: Effect of change in mortality rate with time.

In case of an infinite pool, this problem can be rectified. Referring back to equation (4.1) or (4.2), under an infinite pool the withdrawal strategy of a member is independent of other members. When there is a future change in the mortality level of a member, given that the change can be correctly predicted, equation (4.2) can be solved simply by substituting the updated mortality rate at each future time point.

We illustrate this through a backtest. The Gompetz model is fitted to the German population data for each of the years from 1973 to 2017, a total of 45 years corresponding to the assumed maximum retirement duration. In Figure 4.13 are the fitted model parameters throughout the period.

Next, the one-year mortality rate of an aged 65 is taken from the model fitted to 1973 data, that of an aged 66 taken from the 1974 model and so on, obtaining the stochastic mortality rate of an individual aged 65 to 109 in years 1973 to



(a) Parameter m.

(b) Parameter b.

Figure 4.13: Evolution of mortality model parameters from 1973 to 2017.

2017. Equation (4.2) is then solved by substituting these values of the mortality rate, the result obtained is the constant benefit with the change in mortality level being taken into consideration.

Figure 4.14 shows the effect caused on the remaining wealth in fund and withdrawal amounts w.r.t. age. With the stochastic change in mortality level, a member should withdrawal less at younger ages but more at the advanced ages, in other words deferring the withdrawals, so that the resulting remaining wealth in fund is always higher.

Figure 4.15 gives the result on the new annual benefit amount together with comparison with other scenarios. Comparing to the case that there is no change in the mortality rate w.r.t. age throughout the period (circle marks), the constant benefit becomes lower when the mortality rate changes stochastically with model parameters evolving as in Figure 4.13 (cross marks). The lowered constant benefit amount is the consequence of the improved population mortality level, as reflected by the increasing trend of m displayed in Figure 4.13a, less young members but more old members perish as time evolves, hence less money enters the notional mortality account and less survival gain is received by a surviving member. On the other hand, when the stochastic change in the mortality rate takes place but is not being considered during the establishment of the withdrawal strategy, a member would withdraw too much at each time point, the resulting annual benefit would no longer be constant and be less than expected

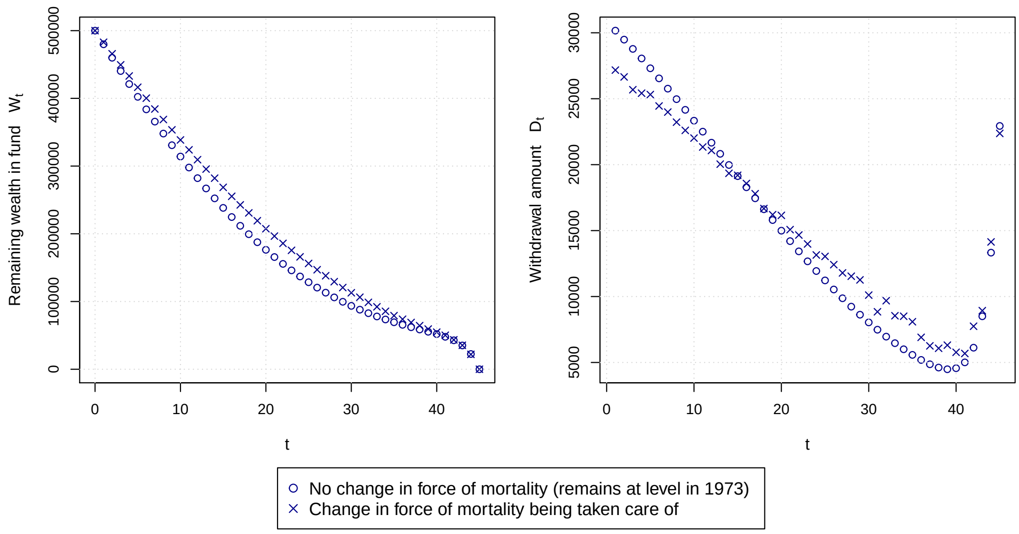


Figure 4.14: Effect on the evolution of the remaining wealth in fund and the withdrawal amount for an infinite pool, when stochastic change in mortality rate is taken into consideration during calculation of the withdrawal strategy.

(triangle marks).

The resulting new annual constant benefit is again comparable to a traditional annuity, where the future change in mortality level is incorporated into the calculation of the annuity price, i.e. adjusting ${}_np_{65}$ in equation (4.22) accordingly.

Needless to say, unlike in the backtest above, in real world the future change in population mortality level is uncertain. In practice, to forecast the future population mortality level, stochastic mortality models are fitted and projections into future are made. When the projected future mortality rate is used for calculating the withdrawal strategy, there exists also the risk on the accuracy of the projection, which is also born by the fund members.

In case of a finite pool, similar adaptation is not possible. Recall that with a finite pool, each member's survival gain is determined also by the wealth-mortality profile of all other members, meaning the withdrawal strategy of a cohort depends on that of the other cohorts. Even leaving out the uncertainty in the change in mortality level, the different evolution of the mortality level for each cohort implies the withdrawal strategy for each cohort must be different. Therefore, the calculations proposed in Section 4.3.2, which is done by implicitly equating the future remaining wealth of one cohort to the current remaining

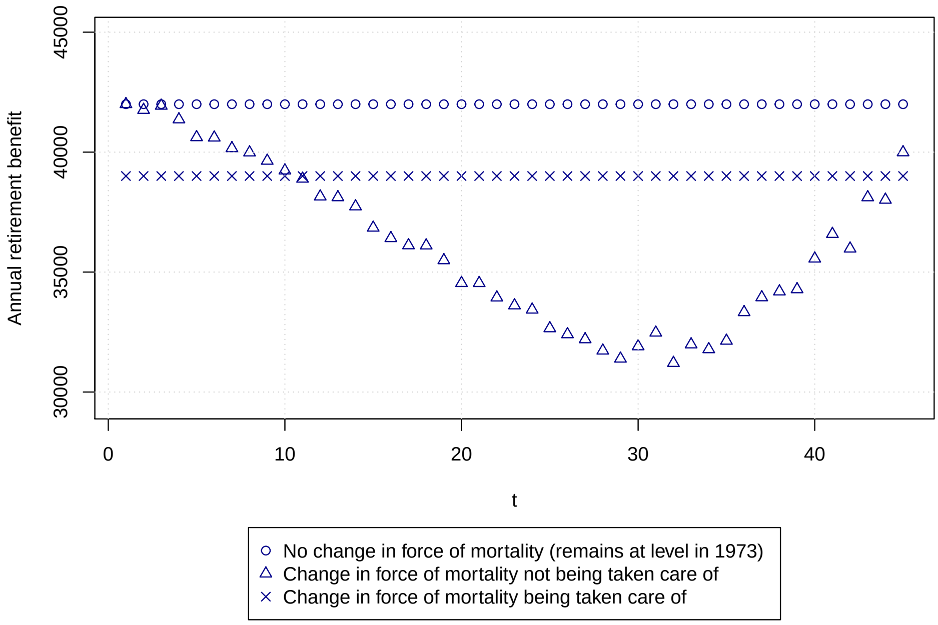


Figure 4.15: Annual benefit in case of an infinite pool, when mortality rate changes stochastically as from 1973 to 2017.

wealth of another cohort, cannot be valid.

4.5 Discussion

The contents in this chapter have demonstrated that, the annuity overlay fund can be operated in a restrictive way, so as to achieve the goal of offering fund members a stream of payment that is constant in expectation over time.

Comparing to a traditional annuity, the annual expected payout from the annuity overlay fund is slightly lower. With a pool of infinite members, where there is no uncertainty in the actual payment amount, our numerical illustration has shown a percentage drop of roughly 1.99% in the retirement benefit from the annuity overlay fund relative to that from an annuity. For a pool with finite members, the members also bear the uncertainty in the actual benefit amount. Further inspection on the figures has suggested that, for a finite pool, the percentage drop in the expected payment from the fund relative to the guaranteed payment from an annuity diminishes to the same magnitude of 1.99% (rounded to two decimal places) already if the fund admits over 175 new retirees yearly. This is a number that is not hard to achieve in practice.

On the other hand, since there is no risk born by the insurer to run the annuity overlay fund, unlike with an annuity, no safety loading is to be added to the price of the fund by the insurer. For an annuity, the safety loadings required by an insurer is usually higher than the percentage suggested in the previous paragraph. For instance, Chen et al. (2019) calculated the risk loadings of an annuity to be as large as 4.84% based on the Solvency II requirements. Hence, after taking the loadings into consideration, the annuity overlay fund is likely to offer an expected benefit being higher than the guaranteed benefit from a traditional annuity. In this case, the annuity overlay fund could be preferred over a traditional annuity, if retirees are willing to bear some level of uncertainty in the retirement benefit payments.

The main advantage of applying the annuity overlay fund in retirement aspect is its feature of being a periodically open fund, which is enabled by the periodically-actuarially-fair property. By permitting new retirees to enter the fund at the beginning of each period, the pool size and thus the variation in payouts can remain relatively stable over time. This is an advantage that product designs requiring a closed pool cannot offer. For example, the tontine which is heavily studied in recent years works only with a closed pool. In terms of retirement application, tontine participants face a greater variation in the benefit amounts to be obtained as they approach advanced ages, when a large portion of members have perished and the pool size becomes tiny (see e.g. Figure 2 in Chen et al. (2019) for the variation of payments from tontine).

An issue of the proposed operation of the annuity overlay fund is that, with a finite pool, it is not possible to incorporate a stochastic mortality rate in the computation of the withdrawal strategy. Nevertheless, our numerical illustrations have demonstrated that, adaptation can be made in case of an infinite pool. Consider also the result that, the expected benefit from a finite pool with merely hundreds of new members yearly already resembles that from an infinite pool. It is therefore proposed that, under the setting of stochastic mortality, the withdrawal strategy calculated from an infinite pool can be used to approximate that for a finite pool with sufficiently many entrants in each period. A more precise elaboration on this possibility, together with investigation on the level of variation faced by members in this case, is left for future research.

It is worth remarking that, in order to apply the annuity overlay fund, it is very important to correctly assign the mortality rate of all fund members with respect to age. An inaccurate assignment of mortality rate would result in a stream of retirement benefit that cannot be constant in expectation over time, analogous to the triangle marks displayed in Figure 4.15. The issue of uncertainty in humans' future mortality rate and its forecasting are an extended research area, which is beyond the scope of the thesis and is not covered here.

Chapter 5

Conclusion

This thesis has contributed in three aspects.

First, a discrete-time model of the annuity overlay fund has been specified. A discrete-time model is often preferred over a continuous-time model by insurance companies for use in practice, the model established in the thesis hence helps narrow down the gap between theoretical investigation and practical application of the annuity overlay fund. Moreover, we have expressed the fund in a slightly different manner comparing to the original design in Donnelly et al. (2014), where the mathematical formulas are simplified and can be more easily comprehended.

Second, analyses based on the risk and return features of the annuity overlay fund have been conducted. By considering the actuarial gain upon survival until the terminal time, it has been found that, older members as well as members who invest less into the fund face a risk-return trade-off at a larger magnitude through engaging in the mortality risk pooling process. The expected change in utility obtained through participating in the risk pooling has been taken as the measure of the fund's attractiveness to an individual, and a framework of analysis has been suggested for examining the relative attractiveness of the fund for members with different wealth-mortality profiles and attitude towards risk.

Lastly, a method to operate the annuity overlay fund has been proposed, with which a stream of payment being constant in expectation over time can be provided to surviving fund members, given that members' mortality rate with respect to age does not change throughout time. This offers individuals another option for pooling mortality risks with others and we advocate that, this can be a new possibility in the retirement aspect. Further research in this regard is encouraged for investigating the practicability of the proposal.

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Appendix A

Notations

	Notation	Meaning
Model		
	T	Terminal time
	M	Number of members in the fund at inception (time 0)
	$W_0^{(k)}$	Wealth invested from member k at time 0
	$W_{T-}^{(k)}$	Wealth of member k at time T before receiving the actuarial gain
	$W_T^{(k)}$	Wealth of member k at time T after receiving the actuarial gain
	${}_TQ_0^{(k)}$	Death probability for member k in the time interval $[0, T]$
	$N_T^{(k)}$	Survival status of member k at time T, 0 means alive and 1 otherwise
	U_T	Total amount flown into the notional mortality account until time T
	$G_T^{(k)}$	Actuarial gain for member k at time T
	$R_T^{(k)}$	Return from financial market for member k at time T
Chapter 3		
	$u_k(W)$	Utility for member k from some wealth W
	$A_k(W)$	Absolute risk aversion coefficient for member k with some wealth W
Chapter 4		
	r	risk-free rate
	q_X	One-year death probability for an individual at age X
	${}_tp_X$	Probability for an individual at age X to survive the next t years

X_r	Retirement age
X_{lim}	Limiting age
t	Duration that a retiree has remained in the fund
$\tau(= X_{lim} - X_r)$	Maximum duration that a retiree remains in the fund
Z	Number of elements in the set of permissible initial wealth amount
$C_z(= C \text{ if } Z = 1)$	Number of members who are admitted into the fund at the beginning of each period, whose initial investment is the z-th element in the set of permissible initial wealth amount
$W_{t,z}(= W_t \text{ if } Z = 1)$	Remaining wealth in the fund for a member at age X_r+t, whose initial investment upon entrance was the z-th element in the set of permissible initial wealth amount
$D_{t,z}(= D_t \text{ if } Z = 1)$	Withdrawal amount for a member at age $X_r + t$, whose initial investment upon entrance was the z-th element in the set of permissible initial wealth amount
$L_{t,z}(= L_t \text{ if } Z = 1)$	Number of members in the fund who are at age $X_r + t$, whose initial investment upon entrance was the z-th element in the set of permissible initial wealth amount

Appendix B

Proof for equation (2.5)

Consider an arbitrary member k of the fund at time 0. The probability of the member perishing during the interval $[0, T]$ is ${}_TQ_0^{(k)}$. Clearly,

$$\mathbb{E}\left(N_T^{(k)} \middle| \mathcal{F}_{T-}\right) = {}_TQ_0^{(k)} \cdot 1 + (1 - {}_TQ_0^{(k)}) \cdot 0 = {}_TQ_0^{(k)}.$$

Conditional on the information at time 0, the expected amount of money that flows into the notional mortality account at time T due to deaths in the time interval $[0, T]$ is

$$\begin{aligned}
\mathbb{E}\left(U_T \middle| \mathcal{F}_{T-}\right) &= \sum_{m=1}^{M} W_{T-}^{(m)} \mathbb{E}\left(N_T^{(m)}\right) \\
&= \sum_{m=1}^{M} W_{T-}^{(m)} {}_TQ_0^{(m)}.
\end{aligned}$$

Next, the actuarial gain at time T, $G_T^{(k)}$, given by equation (2.3) can be written in the compact form

$$G_T^{(k)} = \frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} \; U_T - W_{T-}^{(k)} N_T^{(k)}.$$

Now

$$\begin{aligned}
&\mathbb{E}\left(G_T^{(k)} \middle| \mathcal{F}_{T-}\right) \\
=&\mathbb{E}\left(\frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} \; U_T - W_{T-}^{(k)} N_T^{(k)} \middle| \mathcal{F}_{T-}\right) \\
=&\frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} \; \mathbb{E}\left(U_T | \mathcal{F}_{T-}\right) - W_{T-}^{(k)} \mathbb{E}\left(N_T^{(k)} \middle| \mathcal{F}_{T-}\right) \\
=&{}_TQ_0^{(k)} W_{T-}^{(k)} - {}_TQ_0^{(k)} W_{T-}^{(k)} \\
=&0.
\end{aligned}$$

Appendix C

Proof for equation (3.1)

Consider an arbitrary member k, given that she survives the time interval $[0, T]$, that is, $N_T^{(k)} = 0$, we have

$$\mathbb{E}\left(N_T^{(k)} \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0\right) = 0,$$

and

$$\mathbb{E}\left(U_T \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0\right) = \sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)} - {}_TQ_0^{(k)} W_{T-}^{(k)}.$$

It follows that, the expected actuarial gain for the member conditional on her survival is

$$\begin{aligned}
&\mathbb{E}\left(G_T^{(k)} \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0\right) \\
=&\mathbb{E}\left(\frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}}\ U_T - W_{T-}^{(k)} N_T^{(k)} \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0\right) \\
=&\frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}}\ \mathbb{E}\left(U_T \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0\right) - W_{T-}^{(k)} \mathbb{E}\left(N_T^{(k)} \middle| \mathcal{F}_{T-}, N_T^{(k)} = 0\right) \\
=&\frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}} \left(\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)} - {}_TQ_0^{(k)} W_{T-}^{(k)}\right) - W_{T-}^{(k)} \cdot 0 \\
=&{}_TQ_0^{(k)} W_{T-}^{(k)} \left(1 - \frac{{}_TQ_0^{(k)} W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)} W_{T-}^{(m)}}\right).
\end{aligned}$$

Appendix D

Proof for equation (3.2)

As $N_T^{(k)} = 0$, we have

$$\text{Var}\left(N_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right) = 0$$

and

$$\text{Cov}\left(U_T, N_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right) = 0.$$

Conditional on that the member k survives the time interval $[0, T]$, we have

$$\begin{aligned}\text{Var}\left(U_T\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right) &= \sum_{m=1}^{M}\left(W_{T-}^{(m)}\right)^2 {}_TQ_0^{(m)} - \left(W_{T-}^{(k)}\right)^2 {}_TQ_0^{(k)} \\ &= \sum_{\substack{m=1,\\ m\neq k}}^{M}\left(W_{T-}^{(m)}\right)^2 {}_TQ_0^{(m)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\text{Var}\left(G_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right) \\ =&\text{Var}\left(\frac{{}_TQ_0^{(k)}W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)}W_{T-}^{(m)}}\ U_T - W_{T-}^{(k)}N_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right) \\ =&\frac{{}_TQ_0^{(k)}W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)}W_{T-}^{(m)}}\ \text{Var}\left(U_T|\mathcal{F}_{T-}, N_T^{(k)} = 0\right) - W_{T-}^{(k)}\text{Var}\left(N_T^{(k)}\middle|\mathcal{F}_{T-}, N_T^{(k)} = 0\right) \\ =&\frac{{}_TQ_0^{(k)}W_{T-}^{(k)}}{\sum_{m=1}^{M} {}_TQ_0^{(m)}W_{T-}^{(m)}}\left(\sum_{\substack{m=1,\\ m\neq k}}^{M}\left(W_{T-}^{(m)}\right)^2 {}_TQ_0^{(m)}\right).\end{aligned}$$

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Ich bin mir bewusst, dass eine unwahre Erklärung rechtliche Folgen haben wird.

Ulm, den 07.09.2020

(Unterschrift)